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DYNAMICAL PROBLEMS IN NON-LINEAR ADVECTIVE PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In the first part of the thesis, we consider the initial value problem for the conservative quasi-geostrophic thermal active scalar equation on the 2-dimensional torus,

$$\frac{\partial\theta}{\partial t} + (u\nabla)\theta = 0, \tag{0.1}$$

and a dissipative regularization of (0.1) by a fractional power of Laplacian with a driving force f,

$$\frac{\partial\theta}{\partial t} + (u\nabla)\theta + (-\Delta)^{\alpha}\theta = f, \tag{0.2}$$

where $0 \leqslant \alpha \leqslant 1$. In both cases, the advecting velocity u is related to the scalar θ as

$$u = R^{\perp}\theta = \begin{bmatrix} -R_2\theta \\ R_1\theta \end{bmatrix},$$

R_j being the j-th periodic Riesz transform.

We first interpret the system (0.1) geometrically as the Euler equations on an infinite-dimensional Lie algebra and exhibit the underlying Hamiltonian structure. We prove that for both (0.1) and (0.2) there exist global weak solutions $\theta \in L^{\infty}([0,T]; L^2)$ for each $T > 0$ and initial data $\theta_0 \in L^2$. (For (0.2), an additional natural assumption on f is needed.) The weak solutions of (0.1) exactly preserve the Hamiltonian of the system and satisfy the kinetic energy inequality.

There is an obvious *a priori* bound on the L^p norm of the solution of (0.1). We prove the corresponding result for (0.2), namely,

$$\|\theta(t)\|_p \leqslant \|\theta_0\|_p + \int_0^t \|f(\tau)\|_p \, d\tau,$$

for each $0 \leqslant \alpha \leqslant 1$ and $1 < p \leqslant \infty$.

In particular, for $p = \infty$, this is a maximum principle for the solutions of (0.2).

If $\alpha > \frac{1}{2}$ and $\theta_0 \in L^q$, $f \in L^1([0,T]; L^q)$ with $0 \leqslant \frac{1}{q} < \alpha - \frac{1}{2}$, we prove uniqueness and regularity of solutions of (0.2). (If the force f is sufficiently smooth, then the solution $\theta(t)$ will be at least as smooth as the initial data θ_0.)

These results are proved using the inequality

$$\int |\theta|^{p-2}\, \theta\, (-\Delta)^\alpha \theta\, dx \geqslant 0$$

($1 < p < \infty$, $0 \leqslant \alpha \leqslant 1$), obtained by a probabilistic argument.

A consequence of our results is that in the physically relevant case $\alpha = \frac{1}{2}$ in (0.2) we have global weak solutions on the borderline of known regularity.

In the second part of the thesis, we consider a random advection-diffusion problem on $\mathbb{R}^n$,

$$\frac{\partial \theta}{\partial t} + (u(t,x)\nabla)\theta = \kappa \Delta \theta, \tag{0.3}$$

where the randomness in the velocity field $u(t,x)$ comes from a deterministic dependence on one or several stationary zero-mean Ornstein-Uhlenbeck processes. Using stochastic differential equation representations for (0.3), we prove that the simultaneous N-th moment of the solution, $\left\langle \prod_{j=1}^{N} \theta_j(t,x_j) \right\rangle$, can be obtained by solving another, deterministic equation of passive scalar type in a larger dimensional space.

The application of this technique is to extend an explicit example of turbulent diffusion with white-noise velocity correlation, proposed by A. Majda, to the case of finitely-correlated fields. Similarly to the white-noise case, we obtain moment equations which can be reduced to the parabolic harmonic oscillator problem and solved in quadratures.

This yields exact formulas for moments and higher flatness factors of the distribution of the scalar which agree with the results of Majda in the white-noise limit of the Ornstein-Uhlenbeck process. The long-time asymptotics in the finitely-correlated case are the same as in the white-noise case.

Using the approximation of a stationary Gaussian process by a combination of Ornstein-Uhlenbeck processes, we show that the limiting distribution of the scalar is universal in a larger class of random advecting velocity fields with finite-time correlation. This answers Majda's question about the effect of finite correlation times in the velocity statistics on the distribution of the scalar.

Finally, we show that, in the white-noise limit of the finite-time correlation in the velocity, the moments of the scalar in a more general model asymptotically satisfy Majda's equations. This shows compatibility of the two approaches even when moment equations cannot be solved explicitly.

CHAPTER 1

INTRODUCTION

Many problems in fluid dynamics, including the fundamental Euler and Navier-Stokes equations, have the form of advective evolution equations,

$$\frac{\partial q}{\partial t} + (u\nabla)q = N(q),$$

where q is the unknown quantity (scalar or vector), u is the advecting velocity, and $N(q)$ is a (generally, non-linear) evolution operator, which may describe such effects as scalar dissipation due to viscous friction or diffusion, volume preservation under the flow, vector stretching, or external forces. In the case when the advected quantity q is a scalar field θ, one uses the terms *passive scalar* for a problem in which the velocity field u is given, and *active scalar* for one in which u is determined, usually linearly, from θ.

A typical active scalar problem is a 2-dimensional incompressible flow, in which the velocity u is the perpendicular gradient of a stream function ψ,

$$u = \nabla^{\perp}\psi = \begin{bmatrix} -\partial_2\psi \\ \partial_1\psi \end{bmatrix}.$$

The incompressibility condition $\operatorname{div} u = 0$ is thus automatically satisfied. The stream function ψ is given by an *equation of state* as the convolution of the advected scalar θ with a singular kernel,

$$\psi(x) = \int_{\mathbb{R}^2} K(y)\,\theta(x-y)\,dy \tag{1.1}$$

(see Constantin [11].) Such systems are clearly non-linear; natural examples include 2-dimensional incompressible Euler and Navier-Stokes equations in the vorticity form.

In the passive scalar case, interesting questions arise when the advecting velocity u is a random field with known probability distribution. In a random advection-diffusion problem, the goal is to compute certain average properties ("statistics") of the solution θ of

$$\frac{\partial \theta}{\partial t} + (u(t,x)\nabla)\theta = \kappa \Delta \theta,$$

given appropriate statistics of the velocity u. Although for each fixed realization of u this equation is linear in the unknown function θ, the task of computing statistics of the solution requires the knowledge of the non-linear map which sends realizations of u into corresponding realizations of θ.

In this thesis, we consider examples of both deterministic active scalar and random passive scalar problems. The active scalar example is the thermal quasi-geostrophic (hereafter, QGS) equation studied by Constantin et al. [13, 14] and Held et al. [20, 29] and characterized by the condition that the singular kernel in (1.1) is the Riesz potential of order 1,

$$K(y) = \frac{c_1}{|y|},$$

leading to the equation of state (see Stein [31])

$$\psi = -(-\Delta)^{-\frac{1}{2}}\theta.$$

The mathematical interest in such a non-local problem is that the 2-dimensional conservative QGS equation

$$\frac{\partial \theta}{\partial t} + (u\nabla)\theta = 0 \tag{1.2}$$

is similar to the 3-dimensional Euler equations. Indeed, [13] points out that the evolution equations for the perpendicular gradient of the scalar,

$$\frac{\partial}{\partial t}\nabla^{\perp}\theta + (u\nabla)\nabla^{\perp}\theta = (\nabla u)\,\nabla^{\perp}\theta,$$

have the nonlinearity of same type as the 3-D Euler equations in the vorticity form,

$$\frac{\partial \omega}{\partial t} + (u\nabla)\omega = (\nabla u)\,\omega.$$

Here the advected and stretched vector $\nabla^{\perp}\theta$ corresponds to the Eulerian vorticity ω, the level sets of θ are analogous to the vortex lines in 3-D, and all non-linear terms have the same dimensionality.

One can also consider the dissipative QGS equation with various forms of the dissipation term. According to Majda [23], for the QGS system the dimensionally correct analogue of the kinematic viscosity dissipation in the 3-dimensional Navier-Stokes equations is a term with the square root of (negative) Laplacian,

$$\frac{\partial \theta}{\partial t} + (u\nabla)\theta + (-\Delta)^{\frac{1}{2}}\theta = f.$$

This leads to a broader question about the QGS equation with non-local dissipation terms with general fractional powers of Laplacian $0 \leqslant \alpha \leqslant 1$, i.e.,

$$\frac{\partial \theta}{\partial t} + (u\nabla)\theta + (-\Delta)^{\alpha}\theta = f.$$

We chose to consider the QGS system on the 2-dimensional torus $\mathbb{T}^2$ rather than on the whole plane $\mathbb{R}^2$. This somewhat simplified the existence arguments by providing natural compact embeddings of Sobolev spaces of different orders (Rellich's Theorem), and somewhat complicated the dilation and Littlewood-Paley arguments in Appendix A.2. All qualitative results should be equally applicable to QGS systems on other smooth domains such as the plane $\mathbb{R}^2$ or the sphere $\mathbb{S}^2$.

The random passive scalar part of this thesis concerns a model proposed by Majda [25] to investigate the complex phenomena of turbulent diffusion on a simple example. When a random velocity field has shear geometry and Gaussian distribution, there is a method developed by Avellaneda and Majda [2, 3, 24, 25, 27], which is based on the Feynman-Kac representation of the solutions and allows explicit computation of higher-order moments of the scalar. If the velocity has white-noise correlation in time, the method produces exact equations for the moments [24, 25]. On the other hand, if the velocity has finite correlation time, the moment equations are obtained only in the limit of renormalization rescaling [2]. A special choice of velocity field with white-noise correlation in [25] makes it possible to solve the resulting equations in quadratures and obtain explicit formulas for higher-order moments. This gives

rise to a question about the effect of finite-time correlations in the velocity on the probability distribution of the scalar in a similar model.

We partially settle this question by developing a new technique which produces exact moment equations for a class of finitely-correlated velocity fields including the finite-time analogue of the explicit example in [25]. Similarly to the Feynman-Kac representation approach, our method exploits the dualism between second-order linear parabolic PDEs and stochastic ODEs. As a result, we obtain exact moment equations for velocity fields with randomness coming from deterministic dependence on stationary Ornstein-Uhlenbeck processes. For the Ornstein-Uhlenbeck analogue of [25], it allows explicit computation of moments of the scalar by solving the moment equations in quadratures. The results coincide with [25] both in the long-time and white-noise limits.

We also consider two generalizations of this example. In the first one, we solve the exact moment equations for approximations of a general stationary Gaussian process in the velocity field by a linear combination of independent Ornstein-Uhlenbeck processes. We obtain explicit formulas for the moments on the scalar in these approximations, and show that their long-time asymptotics agree with the previous results. The second approach concerns a more general random shear velocity field with Gaussian distribution and Ornstein-Uhlenbeck covariance. In this case, solving the moment equations explicitly may not be possible, but we are able to show that they are compatible with the Feynman-Kac equations of [24] for the white-noise limit of the Ornstein-Uhlenbeck process.

CHAPTER 2

THE CONSERVATIVE QGS EQUATION

2.1. Preliminaries

We consider the initial value problem for the conservative quasi-geostrophic thermal active scalar equation on the 2-dimensional torus,

$$\frac{\partial \theta}{\partial t} + (u\nabla)\theta = 0, \tag{2.1}$$

with the advecting velocity u related to the scalar θ by

$$u = R^{\perp}\theta = \begin{bmatrix} -R_2\theta \\ R_1\theta \end{bmatrix}, \tag{2.2}$$

R_j being the j-th periodic Riesz transform.

We first introduce the notation. The perpendicular of a vector

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$$

is

$$v^{\perp} \triangleq \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix},$$

and the vector product is

$$v \wedge w \triangleq v^{\perp} w = v_1 w_2 - v_2 w_1.$$

The Fourier series on the torus $\mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$ is

$$f(x) = \sum_{m\in\mathbb{Z}^2} \widehat{f}(m)\, e^{imx}$$

with

$$\widehat{f}(m) = \frac{1}{(2\pi)^2} \int\limits_{\mathbb{T}^2} f(x)\, e^{-imx}\, dx.$$

The inner product in $\mathbb{R}^2$ is implicit where it naturally occurs, as in the exponent above or in the advective term $(u\nabla)\theta$, etc.

We assume that the scalar field has zero average on the torus,

$$\widehat{\theta}(0) = \frac{1}{(2\pi)^2} \int\limits_{\mathbb{T}^2} \theta\, dx = 0. \tag{2.3}$$

This condition is preserved by the time evolution (2.1).

The periodic Riesz transforms in (2.2) are then defined by

$$(R_j\theta)\widehat{\ }(m) = -i\frac{m_j}{|m|}\,\widehat{\theta}(m), \quad m \in \mathbb{Z}^2 \setminus \{0\}.$$

The square root of the (negative) Laplacian is denoted by $\Lambda \triangleq (-\Delta)^{\frac{1}{2}}$, so

$$(\Lambda f)\widehat{\ }(m) = |m|\, \widehat{f}(m).$$

The relation (2.2) can be restated as

$$u = \nabla^{\perp}\psi,$$

where the stream function ψ is related to the scalar θ by the equation of state

$$\psi = -\Lambda^{-1}\theta.$$

A function space without indicated underlying domain (as in L^2) will always refer to the space over $\mathbb{T}^2$. Let H^s be the L^2-based Sobolev space of exponent s, i.e.,

$$H^s = \left\{ f \in \mathscr{D}' : \sum |m|^{2s} \, |\widehat{f}(m)|^2 < \infty \right\}.$$

We will fix an arbitrary time interval $[0,T]$ and consider spaces of Banach-valued functions of time defined over it. The notation will usually be abbreviated to, e.g., $L^\infty(;L^2) \triangleq L^\infty([0,T];L^2(\mathbb{T}^2))$, etc.

The double- and single-bar norms are

$$\|f\|_p \triangleq \|f\|_{L^p},$$
$$|f|_s \triangleq \|f\|_{H^s} = \|\Lambda^s f\|_2.$$

The conservative QGS system possesses a number of integrals of motion. Since velocity defined by (2.2) satisfies $\operatorname{div} u = 0$, it follows that

$$I_\phi = \int_{\mathbb{T}^2} \phi(\theta)\, dx$$

is constant for each smooth function ϕ (cf. (2.3).) In particular, all L^p norms of θ, $1 < p < \infty$, are conserved; hence, by passing to the limit, so are L^1 and L^∞ norms. The L^2 norm of the scalar is proportional to the total kinetic energy of the flow because the vector Riesz transform in (2.2) is an L^2 isometry. Another conserved quantity is

$$H = -\frac{1}{2}\int_{\mathbb{T}^2} \psi\, \theta\, dx,$$

which will be shown in Section 2.2 to be the actual Hamiltonian of the system.

2.2. Geometric Interpretation

In this section we interpret the QGS system (2.1) as the Euler equations on a Lie algebra and exhibit their Hamiltonian structure. The arguments follow Arnold [1] and are all formal. We do not distinguish between genuine Lie groups and their infinite-dimensional analogues, etc.

The motion $t \mapsto \gamma(t)$ of a free particle on a Riemannian manifold is characterized by the Lagrangian

$$L = \frac{1}{2}\|\dot{\gamma}\|^2,$$

so the Euler-Lagrange equations coincide with the geodesic equations for the metric. Therefore, the particle moves along a geodesic, its trajectory being a solution of a system of second-order ODEs in local coordinates.

If the manifold is a Lie group G with a left-invariant Riemannian metric, then the substitution

$$\dot{\gamma}(t) = dL_{\gamma(t)}v(t), \tag{2.4}$$

where L_g is the left multiplication in G, reduces the equations of the geodesic flow to the system of first-order ODEs

$$\dot{v} + B(v, v) = 0 \tag{2.5}$$

on the Lie algebra $\mathfrak{g}$ of the group. Here the bilinear form $B(u, v)$ is defined on $\mathfrak{g}$ by

$$\nabla_{X_u} X_v = X_{B(u,v)}$$

for the left-invariant vector fields $X_v(g) \triangleq dL_g v$ and the Levi-Civita connection $\nabla_X Y$ associated with the metric. These are the Euler equations on a Lie algebra. The standard examples are the Euler equations of the rigid body, with $G = SO_3(\mathbb{R})$, and the Euler equations of the inviscid incompressible fluid in a domain $\Omega \subset \mathbb{R}^n$, with $G = S\mathrm{Diff}(\Omega)$, the group of all volume-preserving diffeomorphisms of Ω. (To consider $S\mathrm{Diff}(\Omega)$ from this *left*-invariant point of view, the group operation has to be defined as $\phi \cdot \psi \triangleq \psi \circ \phi$.) Such Euler equations on a Lie algebra are always quadratic, conservative, and, moreover, form a Hamiltonian system (see [1] for a detailed discussion.)

The substitution (2.4) can be regarded as the passage from the Lagrangian coordinates $\gamma(t) \in G$ of the system to the Eulerian ones $v(t) \in \mathfrak{g}$.

The bilinear form $B(u, v)$ can be defined by the properties

$$(B(u, u), w) = ([w, u], u),$$
$$[u, v] = B(u, v) - B(v, u),$$

which only use the commutator and the inner product on $\mathfrak{g}$. In fact, only the first one of these formulas is needed for (2.5). This makes it possible to define the Euler equations on a Lie algebra with an inner product without mentioning the Lie group itself.