

CLASSIFICATION OF THE SUBALGEBRAS OF THE ALGEBRA OF ALL 2 BY 2
MATRICES

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MATRICES

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JUSTIN LUIS BERNAL

THESIS

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Abstract

Classification of the subalgebras of the familiar algebra of all $n \times n$ real matrices over the real numbers can get quite unwieldy as all subalgebras are of dimension ranging from 1 to n^2 . Classification of the subalgebras of the algebra of all 2×2 real matrices over the real numbers is an interesting first start.

Since $\mathbf{M}_2(\mathbb{R})$ is of dimension 4 then its possible subalgebras are of dimension 1, 2, 3, or 4. The one-dimensional subalgebra and four-dimensional subalgebra need little to no attention. The two-dimensional and three-dimensional subalgebras however turn out to be of significance.

It turns out there is only one one-dimensional subalgebra and one four-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$. The former being fairly simple and the latter being trivial. The investigation of the two-dimensional and three-dimensional subalgebras is not as brief. Therefore, the goal of this thesis is to answer the following question:

Up to an isomorphism, how many distinct two-dimensional and three-dimensional subalgebras of $\mathbf{M}_2(\mathbb{R})$ are there?

We show here that up to an isomorphism there are three distinct two-dimensional subalgebras and one distinct three-dimensional subalgebra.

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PREVIEW

Chapter 1

Introduction

This chapter is intended to ease us into the real-algebra of $n \times n$ real matrices. We begin with some familiar definitions that are immediately called upon in the definition of an algebra over a ring, or later simply called an algebra. Then to better understand isomorphisms between algebras we go over needed homomorphisms. Finally an algebra is defined and then follows that, for a given n , the set of all $n \times n$ real matrices is easily equipped to be a real-algebra.

1.1 Ring Definition

First we adopt a familiar definition of a ring in anticipation of the algebra definition.

Definition 1 *A ring is a set \mathbf{R} together with two binary operations*

$$+ : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$$

and

$$\cdot : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$$

satisfying the following properties (as is customary we write $a + b$ in place of $+(a, b)$ and write ab in place of $\cdot(a, b)$ for all $a, b \in \mathbf{R}$):

- $(\mathbf{R}, +)$ is an abelian group
- (\mathbf{R}, \cdot) is a monoid
- $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in \mathbf{R}$.

If in addition $(\mathbf{R} \setminus \{\text{identity for } +\}, \cdot)$ is an abelian group then \mathbf{R} is called a field.

Naturally a subset $\mathbf{S} \subset \mathbf{R}$ is said to be a subring of \mathbf{R} if $(\mathbf{S}, +)$ is a subgroup of the group $(\mathbf{R}, +)$ and also (\mathbf{S}, \cdot) is a submonoid of the monoid (\mathbf{R}, \cdot) .

It is of particular importance that (\mathbf{R}, \cdot) is a monoid so that a ring indeed contains unity, call it $1 \in \mathbf{R}$.

1.2 Vector Space Definition

Once more in anticipation of the algebra definition, we now define a vector space.

Definition 2 Let \mathbf{F} be a field.

A vector space over \mathbf{F} (or vector space if \mathbf{F} is fixed for a particular discussion) is an abelian group \mathbf{G} (with binary operation $+$) together with a scalar multiplication map

$$\cdot : \mathbf{F} \times \mathbf{G} \rightarrow \mathbf{G}$$

that satisfy the following properties for all $a, b \in \mathbf{F}$ and $m, n \in \mathbf{G}$ (as is customary we write am in place of $\cdot(a, m)$ for all $a \in \mathbf{F}$ and $m \in \mathbf{G}$):

- $a(m + n) = am + an$
- $(a + b)m = am + bm$
- $(ab)m = a(bm)$
- $1m = m$.

A subset $\mathbf{H} \subset \mathbf{G}$ is said to be a vector subspace of \mathbf{G} (or subspace of \mathbf{G} , or a subspace if \mathbf{G} is fixed for a particular discussion) if \mathbf{H} is a subgroup of the group \mathbf{G} that is also closed under the scalar multiplication on \mathbf{G} .