CLASSIFICATION OF THE SUBALGEBRAS OF THE ALGEBRA OF ALL 2 BY 2 MATRICES

JUSTIN LUIS BERNAL

Master's Program in Mathematical Science

APPROVED:

Piotr Wojciechowski, Ph.D., Chair

Emil Daniel Schwab, Ph.D.

Martine Ceberio, Ph.D.

Stephen Crites, Ph.D.
Dean of the Graduate School

to my

FRIENDS and FAMILY

with love

CLASSIFICATION OF THE SUBALGEBRAS OF THE ALGEBRA OF ALL 2 BY 2 MATRICES

by

JUSTIN LUIS BERNAL

THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at El Paso

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

Department of Mathematical Sciences

THE UNIVERSITY OF TEXAS AT EL PASO

May 2020

Abstract

Classification of the subalgebras of the familiar algebra of all $n \times n$ real matrices over the real numbers can get quite unwieldy as all subalgebras are of dimension ranging from 1 to n^2. Classification of the subalgebras of the algebra of all 2×2 real matrices over the real numbers is an interesting first start.

Since $\mathbf{M}_2(\mathbb{R})$ is of dimension 4 then its possible subalgebras are of dimension $1, 2, 3$, or 4. The one-dimensional subalgebra and four-dimensional subalgebra need little to no attention. The two-dimensional and three-dimensional subalgebras however turn out to be of significance.

It turns out there is only one one-dimensional subalgebra and one four-dimensional subalgebra of $\mathbf{M}_2(\mathbb{R})$. The former being fairly simple and the latter being trivial. The investigation of the two-dimensional and three-dimensional subalgebras is not as brief. Therefore, the goal of this thesis is to answer the following question:

> *Up to an isomorphism, how many distinct two-dimensional and three-dimensional subalgebras of* $\mathbf{M}_2(\mathbb{R})$ *are there?*

We show here that up to an isomorphism there are three distinct two-dimensional subalgebras and one distinct three-dimensional subalgebra.

Table of Contents

Chapter 1

Introduction

This chapter is intended to ease us into the real-algebra of $n \times n$ real matrices. We begin with some familiar definitions that are immediately called upon in the definition of an algebra over a ring, or later simply called an algebra. Then to better understand isomorphisms between algebras we go over needed homomorphisms. Finally an algebra is defined and then follows that, for a given n, the set of all $n \times n$ real matrices is easily equipped to be a real-algebra.

1.1 Ring Definition

First we adopt a familiar definition of a ring in anticipation of the algebra definition.

Definition 1 *A ring is a set* $\mathbf{R}$ *together with two binary operations*

$$+ \colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

and

$$\cdot \colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$$

satisfying the following properties (as is customary we write $a + b$ *in place of* $+(a, b)$ *and write* ab *in place of* $\cdot(a, b)$ *for all* $a, b \in \mathbf{R}$*):*

- $(\mathbf{R}, +)$ *is an abelian group*
- $(\mathbf{R}, \cdot)$ *is a monoid*
- $a(b + c) = ab + ac$ *and* $(b + c)a = ba + ca$ *for all* $a, b, c \in \mathbf{R}$.

If in addition $(\mathbf{R}\backslash\{\textit{identity for } +\}, \cdot)$ *is an abelian group then* $\mathbf{R}$ *is called a field.*
Naturally a subset $\mathbf{S} \subset \mathbf{R}$ *is said to be a subring of* $\mathbf{R}$ *if* $(\mathbf{S}, +)$ *is a subgroup of the group* $(\mathbf{R}, +)$ *and also* $(\mathbf{S}, \cdot)$ *is a submonoid of the monoid* $(\mathbf{R}, \cdot)$.

It is of particular importance that $(\mathbf{R}, \cdot)$ is a monoid so that a ring indeed contains unity, call it $1 \in \mathbf{R}$.

1.2 Vector Space Definition

Once more in anticipation of the algebra definition, we now define a vector space.

Definition 2 *Let* $\mathbf{F}$ *be a field.*
A vector space over $\mathbf{F}$ *(or vector space if* $\mathbf{F}$ *is fixed for a particular discussion) is an abelian group* $\mathbf{G}$ *(with binary operation* $+$*) together with a scalar multiplication map*

$$\cdot : \mathbf{F} \times \mathbf{G} \to \mathbf{G}$$

that satisfy the following properties for all $a, b \in \mathbf{F}$ *and* $m, n \in \mathbf{G}$ *(as is customary we write* am *in place of* $\cdot(a, m)$ *for all* $a \in \mathbf{F}$ *and* $m \in \mathbf{G}$*):*

- $a(m + n) = am + an$
- $(a + b)m = am + bm$
- $(ab)m = a(bm)$
- $1m = m$.

A subset $\mathbf{H} \subset \mathbf{G}$ *is said to be a vector subspace of* $\mathbf{G}$ *(or subspace of* $\mathbf{G}$*, or a subspace if* $\mathbf{G}$ *is fixed for a particular discussion) if* $\mathbf{H}$ *is a subgroup of the group* $\mathbf{G}$ *that is also closed under the scalar multiplication on* $\mathbf{G}$.