ON COMPACT AND LOCALLY COMPACT
P-ADIC TOPOLOGICAL ABELIAN GROUPS

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P-ADIC TOPOLOGICAL ABELIAN GROUPS

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Abstract

Compact and locally compact p-adic topological abelian groups are discussed and representation theorems are given for these groups when they are T_o.

<u>Table of Contents</u>

Introduction

In recent years, the concept of a topological abelian group [Definition (1.1)] has played an increasing role in the investigation of the structure of abelian groups.

One of the most useful topologies that has been used in abelian groups is the p-adic topology [Definition (1.12)]. In this topology, it is often quite easy to relate group theoretic and topological concepts, yielding new and often fruitful approaches to classifying abelian groups.

This paper will discuss abelian groups which are compact or locally compact in their p-adic topology and give representation theorems where possible; in fact, Section 2 will give a representation theorem for T_o, compact p-adic abelian groups and Section 3 will do the same for T_o, locally compact p-adic abelian groups.

In order to initiate the discussion, the notation to be used and certain basic facts about topological abelian groups must be set down. This will be done in the following two sections.

0. Notation and Background

The standard reference for results on abelian groups will be Fuchs [2]. The following notation will be used. The notation "$A \subset B$" will mean that the set A is a subset of the set B, while the notation "$H < G$" will mean that the group H is a subgroup of the group G. The first countable ordinal will be written as ω. The direct sum of a finite number, say m, of groups G_i, will be denoted by $\sum_{i=1}^{m} G_i$ or $G_1 + \ldots + G_m$. Otherwise $\sum_{\alpha \in \mathcal{A}} G_\alpha$ will be written for the direct sum of the family of groups $\{G_\alpha : \alpha \in \mathcal{A}\}$. Similarly, $\sum^*_{\alpha \in \mathcal{A}} G_\alpha$ will denote the direct product of the family of groups $\{G_\alpha : \alpha \in \mathcal{A}\}$. The additive group of integers and the additive group of p-adic integers will be written as Z and Z*, respectively, while Q and R will denote the additive group of rational and real numbers, respectively. Finally, $\langle S \rangle$ will be used to denote the group generated by the set S. In particular, if $S = \{a\}$, then $\langle a \rangle$ is the cyclic group generated by the element a.

Topological definitions and facts will be from Kelley [6] while standard results on topological groups will come from Hewitt and Ross [5]. However, the T_0 separation axiom will be made a part of the regularity axiom and hence regular spaces will satisfy Hausdorff's separation axiom. The

closure of a set A will be written as A^-. For a topological space X and a subspace Y, the closure of any subset B of Y, in Y, will be written as B_y^- while B^- will stand for the closure of B in X.

Finally, the symbol "▮" will indicate the end of a proof while the symbol "#" will indicate that a contradiction has arisen.

1. Definitions and Fundamental Properties

(1.1) Definition. A topological abelian group (TAG) is a triple (G, T(G), +) where (G, +) is an abelian group and T(G) is a topology on G for which

(i) the function $\alpha: G \times G \longrightarrow G$ defined by $\alpha(g_1, g_2) = g_1 + g_2$ is continuous, where $G \times G$ has the product topology;

(ii) the function $i: G \longrightarrow G$ defined by $i(g) = -g$ is continuous.

Throughout the remainder of the paper G will be written instead of the triple (G, T(G), +) for a TAG.

The following well-known propositions and corollaries will be needed and their proofs may be found in Hewitt and Ross [5, pp. 16-51].

(1.2) Proposition. Let G be a TAG. Then the translation function $t_g: G \longrightarrow G$ defined by $t_g(x) = x + g$ is a homeomorphism.

(1.3) Corollary. Let G be a TAG. If U is an open set, then for each $g \in G$, $g + U$ is an open set.

(1.4) Proposition. Let G be a TAG and $\mathcal{U}_0$ an open neighborhood base at 0. Then the family $\{g + U_0 : U_0 \in \mathcal{U}_0\}$ is an open neighborhood base at g, for each $g \in G$.