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CLASS FUNCTIONS IN THE GENERAL RADICAL THEORY OF RINGS AND ALGEBRAS

by

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Paul O. Enersen

CONTENTS

INTRODUCTION

The purpose of this dissertation is to continue the study of the general theory of radicals; a theory which was introduced independently by S. A. Amitsur [1,2,3] in 1952 and A. G. Kurosh [15] in 1953 and has been advanced by many others. A summary of much of this development can be found in [9], [11], and [14].

Known results which will be of use to us are presented in Chapter I. However, this chapter should by no means be construed as an exhaustive survey of the literature in this field.

Chapter II is primarily devoted to furthering the radical theory in the class of all not necessarily associative rings. In particular, the class functions U and S are studied, where for any subclass M of a universal class W of rings, $UM = \{R \in W \mid$ no nonzero homomorphic image of R lies in M$\}$ and $SM = \{R \in W \mid$ no nonzero two-sided ideal of R lies in M$\}$. It is well known that UM is a radical class if the class M is s-complete; i.e., if every nonzero ideal of each nonzero ring in M has a nonzero homomorphic image in M. However, the converse fails as shown in two examples in this chapter. In Theorem 2.4 we give necessary and sufficient conditions on a class M in order that UM be a radical class, a hereditary radical class, or a hypernilpotent radical

class. Theorem 2.15 supplies necessary and sufficient conditions on M for SM to be a semisimple class, and Theorem 2.16 gives a similar result in the case that all rings under consideration are associative. The remainder of the chapter studies subclasses which determine the same upper radical class or the same semisimple class.

In Chapter III we direct our attention to classes of algebras over commutative rings. Again the class functions U and S are studied, along with the function L, where LM is the smallest radical class containing a subclass M. Examples are given which show that each of these functions, when restricted to only the algebras in a given class of rings, may yield different classes than when the functions are defined from the outset in terms of algebra ideals and algebra images of the algebras in the class.

Finally, in Chapter IV a construction of Ju. M. Rjabuhin [18] is adapted to study strongly hereditary radical classes; i.e., classes for which the radical of any ideal of a ring coincides with the intersection of the ideal and the radical of the entire ring. Conditions are given under which such a radical class must contain all rings of certain specified types.

Throughout this work, with the exception of Chapter III, the word "ring" will always mean a not necessarily associative ring; associative rings will be designated as such. The reader is assumed to be familiar with the rudiments of ring theory, as found in McCoy [16]. In particular, the

isomorphism theorems will often be used without comment. Since any homomorphic image of a ring R is isomorphic to the factor ring R/I for some ideal I of R, and since all of the classes of rings to be considered are closed under isomorphisms, arbitrary images will be denoted in this form. (Henceforth, the words "image" and "ideal" will be used to mean homomorphic image and two-sided ideal, respectively.)

By a <u>class</u> of rings we shall mean a collection of rings, closed under isomorphisms, which share some common property. If M is a class of rings and $R \in M$, then R is called an <u>M-ring</u>. If a ring R has an ideal I in the class M, we say that I is an <u>M-ideal</u> of R.

Lastly, the symbol $\subseteq$ will be used to denote set containment, whereas the symbol $\subset$ will be reserved for proper containment; i.e., $A \subset B$ if and only if $A \subseteq B$ and $A \neq B$; and $A - B$ will denote set difference.

CHAPTER I

PRELIMINARY RESULTS

In this chapter we present some basic definitions and results from the general theory of radical classes. Only those definitions and results which have direct bearing on the remainder of this work will be stated.

DEFINITION 1.1: Let M be a class of rings.

(1) M is said to be homomorphically closed if every homomorphic image of an M-ring is itself an M-ring.

(2) M is said to be hereditary if every ideal of an M-ring R is an M-ideal of R.

DEFINITION 1.2: A class W of rings is said to be a universal class if W is homomorphically closed and hereditary.

Since all of our work will be carried out in some universal class of rings, it is useful to have examples of such classes at hand.

REMARK 1.3: Each of the following classes of rings is easily seen to be a universal class.

(1) The class of all rings.

(2) The class of all alternative rings; i.e., rings

in which all elements x and y satisfy the relationships $x(xy) = (xx)y$ and $(yx)x = y(xx)$.

(3) The class of all associative rings.

(4) Any class consisting entirely of simple rings together with the trivial ring of only one element. (Henceforth this ring will be denoted by $\underline{(0)}$.)

(5) Any nonempty intersection of two universal classes.

Throughout this dissertation, the symbol $\underline{W}$ will be used to denote an arbitrary universal class of rings. If a universal class is not explicitly mentioned in some discussion, it should be tacitly assumed that we are working in the class W.

Much of our work will be directly concerned with the following class functions; i.e., functions with domain and range the collection of all subclasses of a universal class.

<u>DEFINITION 1.4</u>: Let M be an arbitrary subclass of W. The class functions $\underline{H}$, $\underline{I}$, $\underline{S}$, and $\underline{U}$ are defined as follows.

(1) $\underline{HM} = \{R \in W \mid R = K/I$ for some $K \in M$ and some ideal I of $K\}$. HM is called the <u>homomorphic closure of M</u>.

(2) Let $I_1M = \{R \in W \mid R$ is an ideal of K for some M-ring $K\}$. For each $n = 1,2,\cdots$, let $I_{n+1}M = I_1(I_nM)$. Then we define $\underline{IM} = \bigcup_{n=1}^{\infty} I_nM$. IM is called the <u>hereditary closure of M</u>.

(3) $\underline{SM} = \{R \in W \mid$ if I is a nonzero ideal of R, then $I \not\subset M\}$.

(4) $\underline{UM} = \{R \in W \mid \text{every } (0) \neq R/I \notin M\}$.

The class functions H and I are introduced primarily for the sake of convenience and conservation of space in some of the ensuing arguments. However, the functions U and S will be studied in detail. Before proceding, some of the more basic properties of these functions should be noted.

If M is a subclass of W, a class M* is called the <u>smallest</u> class with property P containing M if $M \subseteq M^*$, M* has property P, and if $M \subseteq N$ where N also has property P, then $M^* \subseteq N$. Note that this is more restrictive than merely saying that M* is a minimal such class.

<u>REMARK 1.5</u>: Let M be a subclass of W. The class functions of Definition 1.4 satisfy the following easily verified properties.

(1) HM and IM are actually independent of the universal class containing M. However, UM and SM depend on the universal class W; i.e., if M is contained in another universal class W*, it may well be that UM in W* is not the same as UM in W, and similarly for SM.

(2) If N is another subclass of W such that $M \subseteq N$, then $HM \subseteq HN$, $IM \subseteq IN$, $UN \subseteq UM$, and $SN \subseteq SM$; i.e., H and I are order preserving, while U and S are order reversing.

(3) The trivial ring (0) is necessarily an element of each of the classes HM, IM, UM, and SM. Furthermore, $UM \cap M \subseteq \{(0)\}$ and $SM \cap M \subseteq \{(0)\}$.

(4) HM is the smallest homomorphically closed

subclass of W containing M. Thus M is homomorphically closed if and only if M = HM.

(5) IM is the smallest hereditary subclass of W which contains M; M is hereditary if and only if M = IM.

We now present the definition around which the theory is centered.

DEFINITION 1.6: A subclass P of W is called a radical class (in W) if it satisfies the following two properties.

(R1) P = HP; i.e., P is homomorphically closed.

(R2) If $R \in W - P$, there exists an ideal I of R such that $(0) \neq R/I \in SP$.

Obviously, the definition of a radical class depends upon the universal class W; a given class P may be a radical class in one universal class containing it, but not in another. Property (R1) guarantees that (0) is in every radical class, so together with part (3) of Remark 1.5, we observe that if P is a radical class, then $P \cap SP = \{(0)\}$. Also, if P is a radical class, SP is called the semisimple class of P.

The next two theorems give necessary and sufficient conditions for a class to be radical.

THEOREM 1.7 [15, page 16]: A subclass P of W is a radical class in W if and only if there exists in every ring $R \in W$ an ideal P(R), called the P-radical of R, satisfying:

(1) $P(R) \in P$.

(2) $P(R)$ contains every P-ideal of R.

(3) $R/P(R) \in SP$.

(4) If $R/Q \in SP$ for some ideal Q of R, then $P(R) \subseteq Q$.

<u>THEOREM 1.8</u> [2, page 103]: A subclass P of W is a radical class in W if and only if each of the following properties holds.

(1) $P = HP$.

(2) For any $R \in W$, the union of an ascending chain of P-ideals of R must also be a P-ideal.

(3) If $R \in W$ and I is an ideal of R such that both $I \in P$ and $R/I \in P$, then $R \in P$. (A class which satisfies this property is said to be <u>closed under homomorphic extensions</u>.)

It is naturally the case that if the classes of rings under consideration are restricted to those which contain only associative rings, stronger results are obtainable. For example, the next theorem and its corollary hold for associative rings, but are not valid in an arbitrary universal class of rings.

<u>THEOREM 1.9</u> [4, page 596]: Let P be a radical class in a universal class V of associative rings. Then for every $R \in V$ and every ideal I of R, the radical $P(I)$ is an ideal of R.

<u>COROLLARY 1.10</u> [4, page 597]: Let P be a radical class

in a universal class V of associative rings. Then for every $R \varepsilon V$ and every ideal I of R, $P(I) \subseteq I \cap P(R)$.

This corollary helps to motivate a definition.

<u>DEFINITION 1.11</u>: Let P be a radical class in W. P is said to be <u>strongly</u> <u>hereditary</u> if for every $R \varepsilon W$, $P(I) = I \cap P(R)$ for all ideals I of R.

The choice of a name for this kind of radical class suggests that strong hereditariness is more restrictive than hereditariness in radical classes. The use of this terminology is justified by the following theorem.

<u>THEOREM 1.12</u>: Let P be a radical class in W.

(1) [7, page 2] If P is strongly hereditary, P is hereditary.

(2) [4, pages 595 and 597] If P is hereditary and W consists only of associative rings, then P is strongly hereditary.

A characterization of strongly hereditary radical classes which will prove to be useful to us is given next.

<u>THEOREM 1.13</u> [4, page 595]: A radical class P in W is strongly hereditary if and only if both P and SP are hereditary.

Somewhat analogous to the study of radical classes is the corresponding study of semisimple classes.