QUALITATIVE AND QUANTITATIVE ANALYSIS OF A FLUID-STRUCTURE INTERACTIVE PARTIAL DIFFERENTIAL EQUATION MODEL

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Matthew Ryan Dvorak

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QUALITATIVE AND QUANTITATIVE ANALYSIS OF A FLUID-STRUCTURE INTERACTIVE PARTIAL DIFFERENTIAL EQUATION MODEL

Matthew Ryan Dvorak, Ph. D.

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Adviser: George Avalos

In this work we consider a coupled partial differential equation (PDE) model which has appeared in the literature to model various fluid-structure interactions seen in nature (see e.g., [13]). It has been recently shown in [2] that this fluid-structure interactive PDE model admits of an *explicit* semigroup generator representation $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$, where $\mathbf{H}$ is the associated space of fluid-structure initial data. In [2], however, the argument for the maximality criterion is indirect, and does not provide for an explicit solution $\Phi \in D(\mathcal{A})$ of the equation $(\lambda I - \mathcal{A})\Phi = F$, for given $F \in \mathbf{H}$ and $\lambda > 0$.

The work in Chapter 1 reconsiders the proof of maximality for the fluid-structure generator $\mathcal{A}$, and gives an explicit method for solving the fluid-structure equation. This methodology involves a nonstandard usage of the Babuška-Brezzi Theorem. Chapter 2 contains a proof of strong stability of the semigroup generated by the fluid-structure operator $\mathcal{A}$. Thence solutions of the fluid-structure interactive PDE are asymptotically stable, see (2.1).

The work in Chapter 3 develops a finite element method for approximating solutions of the fluid-structure system; it is based upon our explicit proof of maximality and does not make use of the divergence-free basis functions usually employed in fluid dynamics. A numerical example involving an eigenfunction of $\mathcal{A}$ is used to test the method.

The final chapter contains a *nonlinear* fluid-dynamics result based upon the methodology developed for the linear fluid-structure model, but utilizing nonlinear semigroup theory. Similar results, though employing a Galerkin method of proof, may be found in [25], [20], and [9] among others. This result can be considered a step towards tackling the fluid-structure problem with nonlinear fluid dynamics.

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Chapter 1

Analysis of a Fluid-Structure Interaction

1.1 Introduction

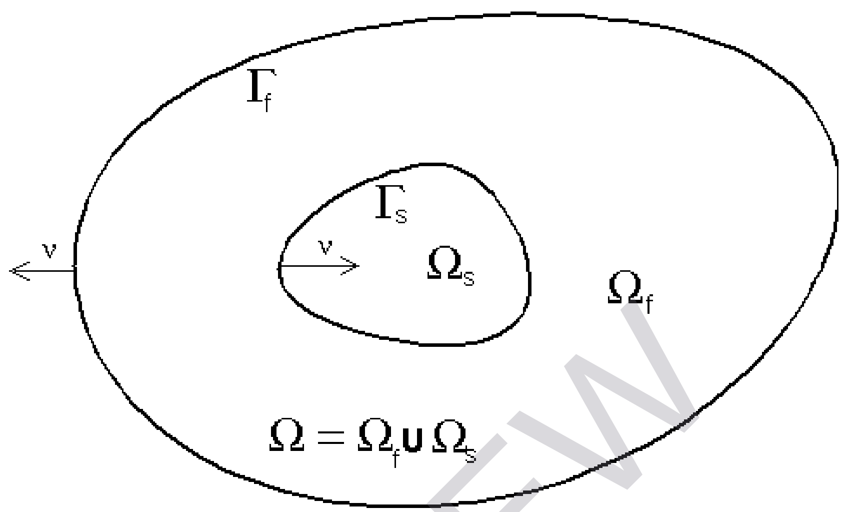
In this work we shall consider a partial differential equation (PDE) system which has been invoked in the existing literature to model various fluid-structure interactions which occur in nature (see e.g., [13] and [8]). Here, we shall deal with the linearized version of the fluid-structure PDE model.

Throughout, $\Omega_f \subseteq \mathbb{R}^n$, $n = 2$ or 3, will denote the bounded domain on which the fluid component of the coupled PDE system evolves with time. The boundary $\partial\Omega_f$ of this domain will be decomposed as $\partial\Omega_f = \Gamma_s \cup \Gamma_f$, $\Gamma_s \cap \Gamma_f = \emptyset$, with each boundary piece being sufficiently smooth. In addition, Ω_s will be the domain on which the structural component evolves with time. The coupling between the two distinct fluid and elastic dynamics occurs precisely because $\partial\Omega_s = \Gamma_s$; see Figure 1. That is to say, Γ_s will serve as a boundary interface on which certain (to be specified) boundary transmission conditions will exert a strong coupling between the Stokes flow in Ω_f and the elastic dynamics which are displacing within Ω_s.

Also, as depicted in Figure 1, $\nu(x)$ will always denote the unit normal vector, which is *exterior* to Ω_f, and so *interior* with respect to Ω_s. (This point will be important to bear in mind, as the direction of ν will influence the computations below.)

With the aforesaid geometrical notions in place, we now proceed to properly introduce the fluid-structure PDE model of our present concern. In dependent variables $u = [u_1(t,x), u_2(t,x), ..., u_n(t,x)]^T$ (the fluid velocity field), $p(t,x)$ (the scalar-valued pressure function), and $w = [w_1(t,x), w_2(t,x), ..., w_n(t,x)]^T$ (the structural displace-

Figure 1.1: The Fluid-Structure Domain



ment field), the following fluid-structure PDE model shall be considered:

$$\text{(PDE)} \quad \begin{cases} u_t - \Delta u + \nabla p = 0 & \text{in } (0,T) \times \Omega_f \\ \operatorname{div}(u) = 0 & \text{in } (0,T) \times \Omega_f \ ; \\ w_{tt} - \Delta w + w = 0 & \text{in } (0,T) \times \Omega_s \end{cases} \tag{1.1}$$

$$\text{(BC)} \quad \begin{cases} u|_{\Gamma_f} = 0 & \text{on } (0,T) \times \Gamma_f \\ u = w_t & \text{on } (0,T) \times \Gamma_s \\ \dfrac{\partial u}{\partial \nu} - \dfrac{\partial w}{\partial \nu} = p\nu & \text{on } (0,T) \times \Gamma_s \end{cases} \tag{1.2}$$

$$\text{(IC)} \quad [w(0,\cdot), w_t(0,\cdot), u(0,\cdot)]^T = [w_0, w_1, u_0]^T \in \mathbf{H}, \tag{1.3}$$

with space of wellposedness

$$\mathbf{H} \equiv [H^1(\Omega_s)]^n \times [L^2(\Omega_s)]^n \times \mathcal{H}_f, \tag{1.4}$$

and with the fluid component space $\mathcal{H}_f \subset [L^2(\Omega_f)]^n$ being defined as follows:

$$\mathcal{H}_f = \{f \in L^2(\Omega_f) : \operatorname{div}(f) = 0 \text{ in } \Omega_f \text{ and } [f \cdot \nu]_{\Gamma_f} = 0\}. \tag{1.5}$$

(Recall that if $f \in [L^2(\Omega_f)]^n$ and $\operatorname{div}(f) \in L^2(\Omega_f)$, one has $[f \cdot \nu]_{\partial\Omega_f} \in H^{-\frac{1}{2}}(\partial\Omega_f)$, and so $\mathcal{H}_f$ is well-defined (see [6], p. 5.). $\mathbf{H}$ is a Hilbert space with the following norm inducing inner product:

$$\left(\begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix}, \begin{bmatrix} \tilde{v}_1 \\ \tilde{v}_2 \\ \tilde{f} \end{bmatrix} \right)_{\mathbf{H}} = (\nabla v_1, \nabla \tilde{v}_1)_{\Omega_s} + (v_1, \tilde{v}_1)_{\Omega_s} + (v_2, \tilde{v}_2)_{\Omega_s} + \left(f, \tilde{f}\right)_{\Omega_f} \tag{1.6}$$

(here of course, $(f, g)_\Omega \equiv \int_\Omega fg d\Omega$).

The objective here will be to ascertain wellposedness of the PDE model (1.1)-(1.3), for initial data $[w_0, w_1, u_0]^T$ in the apparently "natural" space $\mathbf{H}$ of finite energy. Here at the start, we should emphasize that even for this purely linear problem, wellposedness of the dynamics is far from a pedestrian exercise. In fact, the primary difficulty, and the difficulty which will drive our methodology below, lies in an appropriate elimination of the pressure term $p(t, x)$. In classical Navier-Stokes Theory, which involves *uncoupled* fluid flow, such pressure elimination is accomplished by means of the famed Leray (or Helmholtz) projector (see [6]). However, a legitimate application of the Leray projector is based upon a presupposition that the fluid velocity field satisfies the so-called "no slip" boundary condition, i.e. $u|_{\partial\Omega_f} = 0$. In our present

situation, the no slip boundary conditions will certainly *not* be in play, inasmuch as the fluid component of (1.1)-(1.3) is coupled to the structural component via the boundary interface Γ_s. Thus, the use of the Leray Projector is wholly invalid here; consequently, the pressure term in our fluid-structure interaction must be eliminated by nonstandard means.

In this connection, we should note the work in [4], which deals with linear and nonlinear versions of the fluid-structure system (1.1)-(1.3). In [4], an elimination of the pressure, which again is a *sine qua non* for the resolution of the fluid-structure dynamics, is obtained by equating (1.1)-(1.3) with an appropriate variational relation. Thus in [4], issues of wellposedness (and of regularity) for (1.1)-(1.3) are subsequently considered within the context of an associated variational relation, or *weak* form.

On the other hand, in [2], the pressure is eliminated by the very different means of identifying the pressure term $p(t, x)$ in (1.1)-(1.3) as the solution of a certain elliptic boundary value problem (BVP). This BVP contains forcing interior and boundary terms comprised of the fluid and structure variables u and w, as well as (boundary traces) of their derivatives. By writing out $p(t, x)$ as the solution of said BVP (through relevant Green's operators), the authors in [2] are able to give an *explicit* semigroup generator representation for the fluid-structure dynamics (1.1)-(1.3). (And of course, having in hand such a fluid-structure generator, one can, in principle, attempt to glean useful qualitative information for solutions of (1.1)-(1.3); e.g., spectral results for the generator, stability of the semigroup, observability inequalities which are dual to certain boundary controllability problems, and so on.)

A portion of the present work is essentially a revisiting of the wellposedness work in [2]: As we noted, the nonstandard elimination of the pressure term p in [2], by its association with an appropriate elliptic BVP, eventually allows for an explicit semigroup generator formulation of the fluid-structure model (1.1)-(1.3). In fact, the

generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ is given explicitly below in (1.18).

It is justified in [2] that fluid-structure generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ is maximal dissipative, and so by an classical invocation of the Lumer Phillips Theorem, the fluid-structure operator $\mathcal{A}$ generates a contraction semigroup on $\mathbf{H}$. The proof, however, of maximality in [2] is given by an *indirect* argument. Namely, maximality is inferred by deriving a necessary upper bound for the resolvent operator $\mathcal{R}(\lambda; \mathcal{A})$, where $\lambda > 0$, from which the range condition $Range(\lambda I - \mathcal{A}) = \mathbf{H}$ can subsequently be deduced by classical functional analysis; see e.g., Theorem 1.2 of [16].

Thus, although the work in [2] justifies the maximality condition $Range(\lambda I - \mathcal{A}) = \mathbf{H}$, for $\lambda > 0$, the following is *not* explicitly addressed: Given arbitrary fluid-structure data $[v_1^*, v_2^*, f^*]^T \in \mathbf{H}$, how can one find an element $[v_1, v_2, f]^T \in D(\mathcal{A})$ (to be specified below) which solves the static fluid-structure PDE,

$$(\lambda I - \mathcal{A}) \begin{bmatrix} v_1 \\ v_2 \\ f \end{bmatrix} = \begin{bmatrix} v_1^* \\ v_2^* \\ f^* \end{bmatrix} ? \tag{1.7}$$

(Note that from the underlying dissipativity of the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$, as given below in (1.18), the solution $[v_1, v_2, f]^T$ to the equation above will be unique).

Accordingly, this work is partially devoted to giving an explicit methodology for solving the abstract equation (1.7). This will involve a nonstandard usage of the Babuška-Brezzi Theorem, as we shall see below.

As an immediate implication of our novel maximality argument, one can subsequently devise a finite element method (FEM) by which to approximate solutions to the fluid-structure PDE model (1.1)-(1.3). This will be illustrated in the next chapter, within the context of a particular static example. (And of course having in hand a

systematic way to solve (1.7), one can subsequently proceed to solve the time evolving system (1.1)-(1.3), for Cauchy data $[w_0, w_1, u_0]^T \in \mathbf{H}$, by invoking the exponential formula for $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$, and corresponding implicit schemes; see e.g., [15].) We should also emphasize here that due to our nonstandard invocation of Babuška-Brezzi in the course of solving (1.7), the fluid basis functions, adopted in the FEM for numerically approximating the solution of (1.7), will *not* need to be divergence free.

1.2 Elimination of the Pressure

As stated, a successful resolution of the fluid-structure system (1.1)-(1.3) depends upon an appropriate elimination of the pressure, an elimination which cannot involve the Leray Projector since the fluid velocity is *not* everywhere zero on the boundary. To this end, we will maneuver as in [2], to identify the pressure term $p(t,\cdot)$ as the solution of a certain elliptic BVP. (We might also note that, at least formally, the association of pressure functions with elliptic BVP's, at least in the context of *uncoupled* fluid flow problems, has been long known, but generally not exploited; see e.g., [7].)

Because of the overarching importance of these steps to eliminate the pressure, we will include them here, rather than merely appealing directly to [2]; otherwise the reader would have absolutely no insight as to why the fluid-structure generator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ assumes the appearance it does in (1.18).

The elimination of the pressure is based upon the following observation: For fixed $t \in (0,T)$, the scalar-valued function $p(t,x)$ solves the following elliptic BVP:

$$\Delta p = 0 \text{ in } \Omega_f \tag{1.8}$$

$$p = \frac{\partial u}{\partial \nu}\cdot\nu - \frac{\partial w}{\partial \nu}\cdot\nu \text{ on } \Gamma_s \tag{1.9}$$

$$\frac{\partial p}{\partial \nu} = (\Delta u)\cdot\nu \text{ on } \Gamma_f. \tag{1.10}$$

This BVP is derived, point-wise in time, in the following way: (i) Taking the divergence of both sides of the fluid PDE in (1.1), and using $\operatorname{div}(u) = 0$, one has (1.8); (ii) Moreover, the expression (1.9) on the boundary interface Γ_s is attained by taking the dot product of both sides of the Neumann boundary condition in (1.2), with the unit normal vector $\nu(x)$. (iii) Finally, the boundary condition (1.10) is obtained by taking the dot product of both sides of the fluid PDE with respect to an appropriate extension of the normal vector $\nu(x)$, and the restricting the resulting quantity to Γ_f (implicitly using the fact that $[u \cdot \nu]_{\Gamma_f} = 0$). We now proceed to "solve" this system (1.8)-(1.10), by means of abstract Green's maps which account for the contribution of boundary data (see e.g., the many references in [12], wherein this classical idea first germinated in PDE control theory): Let the respective Dirichlet and Neumann maps $D_s : L^2(\Gamma_s) \to L^2(\Omega_f)$, $N_f : L^2(\Gamma_f) \to L^2(\Omega_f)$ be given by:

$$h = D_s(g) \iff \begin{cases} \Delta h = 0 & \text{in } \Omega_f \\ h = g & \text{on } \Gamma_s \\ \dfrac{\partial h}{\partial \nu} = 0 & \text{on } \Gamma_f \end{cases} \tag{1.11}$$

$$h = N_f(g) \iff \begin{cases} \Delta h = 0 & \text{in } \Omega_f \\ h = 0 & \text{on } \Gamma_s \\ \dfrac{\partial h}{\partial \nu} = g & \text{on } \Gamma_f \end{cases} \tag{1.12}$$

(and so each map gives rise to a harmonic extension of boundary data). Then by elliptic regularity (see e.g., [14]), the maps satisfy, for all real r:

$$D_s \in \mathcal{L}(H^r(\Gamma_s), H^{r+\frac{1}{2}}(\Omega_f)); \;\; N_f \in \mathcal{L}(H^r(\Gamma_f), H^{r+\frac{3}{2}}(\Omega_f)).$$

With this operator theoretic machinery in hand, the solution of (1.8)-(1.10) can then be written, for fixed $t \in (0,T)$, as

$$p(t) = D_s\left[\left(\frac{\partial u(t)}{\partial \nu}\cdot\nu - \frac{\partial w(t)}{\partial \nu}\cdot\nu\right)_{\Gamma_s}\right] + N_f[(\Delta u(t)\cdot\nu)_{\Gamma_f}] \text{ in } \Omega_f. \tag{1.13}$$

If we now define the linear maps G_1 and G_2 via

$$G_1 w \equiv \nabla\left(D_s\left[\left(\frac{\partial w}{\partial \nu}\cdot\nu\right)_{\Gamma_s}\right]\right) \tag{1.14}$$

$$G_2 u \equiv -\nabla\left(D_s\left[\left(\frac{\partial u}{\partial \nu}\cdot\nu\right)_{\Gamma_s}\right] + N_f[(\Delta u\cdot\nu)_{\Gamma_f}]\right), \tag{1.15}$$

then these and the expression in (1.13) allow one to write the fluid PDE component of the system (1.1)-(1.3) in terms of u and w alone: That is, pressure is eliminated from the Stokes equation in (1.1)-(1.3), so as to have

$$u_t = \Delta u + G_1 w + G_2 u \quad \text{in } (0,T)\times\Omega_f \tag{1.16}$$

1.3 The Fluid-Structure Generator and its Domain of Definition

1.3.1 The Explicit Form of the Generator

Owing to the maps G_i, defined in (1.14) and (1.15), which allow the fluid flow component of (1.1)-(1.3) to be rewritten as the pressure-free equation (1.16), we can now construct a linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ which can be used to abstractly

model (1.1)-(1.3): To wit, the fluid-structure system (1.1)-(1.3) may be written as

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} w \\ w_t \\ u \end{bmatrix} &= \mathcal{A}\begin{bmatrix} w \\ w_t \\ u \end{bmatrix}, \\ [w(0), w_t(0), u(0)]^T &= [w_0, w_1, u_0]^T \in \mathbf{H}, \end{aligned} \tag{1.17}$$

where

$$\mathcal{A} \equiv \begin{bmatrix} 0 & I & 0 \\ \Delta - I & 0 & 0 \\ G_1 & 0 & \Delta + G_2 \end{bmatrix}. \tag{1.18}$$

Of course, the domain of definition, $D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$, must also be specified here. In addition, and much more to the point: If $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ is to generate a semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0} \subset \mathcal{L}(\mathbf{H})$ - and so the solution $[w(t), w_t(t), u(t)]^T$ to (1.1)-(1.3) is obtained by applying $e^{\mathcal{A}t}$ to initial data $[w_0, w_1, u_0]^T$ - then the domain $D(\mathcal{A})$ should be constructed so as to allow for the existence of just such a semigroup. In particular, if $\mathcal{A} : \mathbf{H} \to \mathbf{H}$ is maximal dissipative with respect to the specified domain $D(\mathcal{A})$, then $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0} \subset \mathcal{L}(\mathbf{H})$ would exist as a C_0- contraction semigroup, by the Lumer Phillips Theorem.

Therefore, an appropriate definition of $D(\mathcal{A})$ is all important here. Our statement concerning the $D(\mathcal{A})$ will be explicit and outright. This is in contrast to what was done in [2], wherein the $Range(A)$ is *first* carefully characterized[1], and *then* the domain $D(\mathcal{A})$, with all its intrinsic features, is given via the relation, $D(\mathcal{A}) = \mathcal{A}^{-1}(Range(\mathcal{A}))$; see Theorem 2.1 of [2]. (We should also make mention of the paper [3], in which a generator representation is derived for a more compli-

[1] In fact, it is shown in [2] that $Range(\mathcal{A}) = \left\{[g_0, g_1, f_0]^T \in \mathbf{H} : \int_{\Gamma_s} g_0 \cdot \nu d\Gamma_s = 0\right\}$.

cated fluid-structure system, involving Stokes flow and the Lamé system of elasticity, with the generator domain also being explicitly identified.)

Before stating $D(\mathcal{A})$ explicitly, we first need some preliminaries.

Proposition 1.3.1. *Suppose a $L^2(\Omega_f)$-function ρ satisfies $\Delta\rho \in L^2(\Omega_f)$. Then one has the following boundary trace estimate:*

$$\left\| \rho|_{\partial\Omega_f} \right\|_{H^{-\frac{1}{2}}(\partial\Omega_f)} + \left\| \frac{\partial\rho}{\partial\nu}\Big|_{\partial\Omega_f} \right\|_{H^{-\frac{3}{2}}(\partial\Omega_f)} \leq C\left\{ \|\rho\|_{L^2(\Omega_f)} + \|\Delta\rho\|_{L^2(\Omega_f)} \right\}. \tag{1.19}$$

Proof of Proposition 1.3.1. Since the Sobolev trace map

$$\gamma \in \mathcal{L}(H^2(\Omega_f), H^{\frac{3}{2}}(\partial\Omega_f) \times H^{\frac{1}{2}}(\partial\Omega_f))$$

is surjective, where $\gamma f = \left[f|_{\partial\Omega_f}, \frac{\partial f}{\partial\nu}|_{\partial\Omega_f} \right]$ for $f \in C^\infty(\bar{\Omega}_f)$, there exists a continuous right inverse $\gamma^+ \in \mathcal{L}(H^{\frac{3}{2}}(\partial\Omega_f) \times H^{\frac{1}{2}}(\partial\Omega_f), H^2(\Omega_f))$. That is, $\gamma\gamma^+([\phi_1, \phi_2]) = [\phi_1, \phi_2]$ for all $[\phi_1, \phi_2] \in H^{\frac{3}{2}}(\partial\Omega_f) \times H^{\frac{1}{2}}(\partial\Omega_f)$. Thus, by Green's Theorem we have for ρ (initially smooth enough) and any $[\phi_1, \phi_2] \in H^{\frac{3}{2}}(\partial\Omega_f) \times H^{\frac{1}{2}}(\partial\Omega_f)$,

$$\begin{aligned}
\int_{\partial\Omega_f} \rho\phi_2 d\partial\Omega_f &= \int_{\Omega_f} \rho\Delta\gamma^+([\phi_1, \phi_2])d\Omega_f + \int_{\Omega_f} \nabla\rho \cdot \nabla\gamma^+([\phi_1, \phi_2])d\Omega_f \\
&= \int_{\Omega_f} \rho\Delta\gamma^+([\phi_1, \phi_2])d\Omega_f - \int_{\Omega_f} \Delta\rho\gamma^+([\phi_1, \phi_2])d\Omega_f + \int_{\partial\Omega_f} \frac{\partial\rho}{\partial\nu}\phi_1 d\partial\Omega_f,
\end{aligned}$$

whence we obtain the estimate (1.19), for smooth ρ. A density argument concludes the proof. $\square$

Lemma 1.3.2. *Suppose a pair $(\mu, \rho) \in [H^1(\Omega_f)]^n \times L^2(\Omega_f)$ satisfy the following properties:*

(i) $div(\mu) = 0$;

(ii) $-\Delta\mu + \nabla\rho = h$, *where* $h \in [L^2(\Omega_f)]^n$ *and* $div(h) = 0$.

Then one has the additional boundary regularity for the pair (μ, ρ)*:*

$$
\begin{aligned}
\rho|_{\partial\Omega_f} &\in H^{-\frac{1}{2}}(\partial\Omega_f); \quad \left.\frac{\partial\rho}{\partial\nu}\right|_{\partial\Omega_f} \in H^{-\frac{3}{2}}(\partial\Omega_f);\\
\left.\frac{\partial\mu}{\partial\nu}\right|_{\partial\Omega_f} &\in [H^{-\frac{1}{2}}(\partial\Omega_f)]^n; \quad [(\Delta\mu)\cdot\nu]_{\partial\Omega_f} \in H^{-\frac{3}{2}}(\partial\Omega_f).
\end{aligned}
$$

Proof of Lemma 1.3.2: From the assumption in (ii) we infer that the L^2-function ρ is harmonic: in fact, taking the divergence of both sides of the equation in (ii), we have

$$\Delta\rho = \operatorname{div}(\Delta\mu) + \operatorname{div}(h) = 0,$$

since μ and the forcing term h are each solenoidal. Of course, this equality, as well as others in this argument, are to be understood in the weak derivative or distributional sense. Consequently, the Proposition 1.3.1 provides a meaning (continuously) to the following boundary traces:

$$\rho|_{\partial\Omega_f} \in H^{-\frac{1}{2}}(\partial\Omega_f); \left.\frac{\partial\rho}{\partial\nu}\right|_{\partial\Omega_f} \in H^{-\frac{3}{2}}(\partial\Omega_f). \tag{1.20}$$

In turn, the given Dirichlet trace for square integrable ρ, combined with an application of Green's formula, yields that $\nabla\rho \in [(H^1(\Omega_f))^n]'$. Therefore, the variable $\mu \in [H^1(\Omega_f)]^n$ satisfies the elliptic equation,

$$-\Delta\mu = -\nabla\rho + h \in [\big(H^1(\Omega_f)\big)^n]'.$$

Appealing then to elliptic theory (see e.g., p. 71, Theorem 3.8.1 of [10]), we have that

continuously,

$$\left.\frac{\partial \mu}{\partial \nu}\right|_{\partial\Omega_f} \in [H^{-\frac{1}{2}}(\partial\Omega_f)]^n. \tag{1.21}$$

Finally, taking the dot product of both sides of the equation in (ii) with an appropriate extension of the normal vector, and restricting the resulting relation to $\partial\Omega_f$, we obtain

$$[(\Delta\mu)\cdot\nu]_{\partial\Omega_f} = \left.\frac{\partial \rho}{\partial \nu}\right|_{\partial\Omega_f} - [h\cdot\nu]_{\partial\Omega_f} \in H^{-\frac{3}{2}}(\Omega_f) \tag{1.22}$$

(implicitly, we are also using the fact that since $h \in [L^2(\Omega_f)]^3$ and $\text{div}(h) = 0$, then $[h\cdot\nu]_{\partial\Omega_f}$ is well-defined as an element in $H^{-\frac{1}{2}}(\partial\Omega_f)$; see Proposition 1.4 of [6]). The procurement of (1.20), (1.21) and (1.22) now complete the proof of Lemma 1.3.2. □

1.3.2 Domain of $\mathcal{A} : \mathbf{H} \to \mathbf{H}$

We are now in a position to set the domain of the operator $\mathcal{A} : \mathbf{H} \to \mathbf{H}$, as it is defined in (1.18):

The subspace $D(\mathcal{A})$ is composed of all $[w_0, w_1, u_0]^T \in \mathbf{H}$ which satisfy the following:

(A.1) $[w_0, w_1, u_0]^T \in [H^1(\Omega_s)]^n \times [H^1(\Omega_s)]^n \times ([H^1(\Omega_f)]^n \cap \mathcal{H}_f)$.

(A.2) On the boundary portion Γ_f, the fluid component $u_0|_{\Gamma_f} = 0$.

(A.3) The structural component w_0 satisfies $\Delta w_0 \in [L^2(\Omega_s)]^n$. (So by elliptic theory $\frac{\partial w_0}{\partial \nu}\big|_{\Gamma_s}$ is well-defined as an element of $[H^{-\frac{1}{2}}(\Gamma_s)]^n$; see e.g., p. 71, Theorem 3.8.1 of [10].)

(A.4) The components obey the following relation on the boundary interface Γ_s:

$$u_0 = w_1 \quad \text{on } \Gamma_s. \tag{1.23}$$

(A.5) For the given data $[w_0, w_1, u_0]^T$, there exists a corresponding "pressure" function $\pi_0 \in L^2(\Omega_f)$ such that:

(A.5a) The pair (u_0, π_0) satisfies

$$-\Delta u_0 + \nabla \pi_0 \in \mathcal{H}_f. \tag{1.24}$$

Thus we have, continuously, by Lemma 1.3.2,

$$\begin{aligned} \pi_0|_{\partial\Omega_f} &\in H^{-\frac{1}{2}}(\partial\Omega_f); \quad \left.\frac{\partial \pi_0}{\partial \nu}\right|_{\partial\Omega_f} \in H^{-\frac{3}{2}}(\partial\Omega_f); \\ \left.\frac{\partial u_0}{\partial \nu}\right|_{\partial\Omega_f} &\in [H^{-\frac{1}{2}}(\partial\Omega_f)]^n; [(\Delta u_0)\cdot\nu]_{\partial\Omega_f} \in H^{-\frac{3}{2}}(\partial\Omega_f). \end{aligned} \tag{1.25}$$

(A.5b) The components $[u_0, w_0]$ and associated pressure function π_0 obey the following relation on the boundary interface Γ_s:

$$\frac{\partial u_0}{\partial \nu} - \frac{\partial w_0}{\partial \nu} = \pi_0 \nu \quad \text{on } \Gamma_s. \tag{1.26}$$

Note that, as we showed outright in deriving the BVP (1.8)-(1.10), the function π_0 *a fortiori* satisfies the following BVP:

$$\Delta \pi_0 = 0 \quad \text{in } \Omega_f \tag{1.27}$$

$$\frac{\partial \pi_0}{\partial \nu} = (\Delta u)\cdot\nu \quad \text{on } \Gamma_f \tag{1.28}$$

$$\pi_0 = \left(\frac{\partial u_0}{\partial \nu} - \frac{\partial w_0}{\partial \nu}\right)\cdot\nu \quad \text{on } \Gamma_s. \tag{1.29}$$

Consequently, we can then use the elliptic maps D_s and N_f, defined respectively in (1.11) and (1.12), to identify this pressure π_0 associated with the triple $[w_0, w_1, u_0]^T$,