SECOND ORDER DYNAMIC EQUATIONS ON TIME SCALES

by

Jacob Weiss

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfillment of Requirements

For the Degree of Doctorate of Philosophy

Major: Mathematics

Under the Supervision of Professors Lynn Erbe and Allan Peterson

Lincoln, Nebraska

Dec, 2007

UMI Number: 3284240

UMI®

UMI Microform 3284240

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

SECOND ORDER DYNAMIC EQUATIONS ON TIME SCALES

Jacob Weiss, Ph.D.

University of Nebraska, 2007

Advisors: Lynn Erbe and Allan Peterson

Abstract:

In this paper I will give criteria under which boundary value problems for second order dynamic equations on time scales have positive solutions. There are sections on existance of positive solutions to second-order singular boundary value problems and second-order functional boundary value problems. Also, I will give condtions under wich the second-order formally self-adjoint operator

$$Lx = \left(p(t)x^{\Delta}\right)^{\nabla} + q(t)x = 0$$

is limit-point.

Contents

Chapter 1

Introduction

1.1 Basic Definitions

The calculus of time scales was introduced by Stefan Hilger in [19] under the direction of Bernd Aulbach. A *time scale*, $\mathbb{T}$, is a closed subset of the real numbers. Time scales forms a bridge between discrete and continuous calculus. By the time scale interval $[a, b]$ we mean the real interval $[a, b]$ intersected with $\mathbb{T}$. Open and half-open intervals are handled in the same way. First, we will define the forward and backward jump operators on $\mathbb{T}$.

Definition 1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the *forward jump operator*, $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

If $\{s \in \mathbb{T} : s > t\} = \emptyset$ (i.e. if $t = \max \mathbb{T}$), we take $\sigma(t) = t$.

Similarly, we define the *backward jump operator*, $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

and, if $\{s \in \mathbb{T} : s < t\} = \emptyset$ (i.e. if $t = \min \mathbb{T}$), we take $\rho(t) = t$.

If $f : \mathbb{T} \to \mathbb{R}$ then $f^\sigma(t)$ is understood to mean $f(\sigma(t))$ and $f^\rho(t)$ is understood to mean $f(\rho(t))$.

If $\sigma(t) > t$, we say t is *right-scattered.* If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, we say t is *right-dense.* Similarly, if $\rho(t) < t$, we say t is *left-scattered.* If $t > \inf \mathbb{T}$ and $\rho(t) = t$, we say t is *left-dense.* Points that are both right-dense and left-dense are called *dense.* Points that are both right-scattered and left-scattered are called *isolated.* Also, a left-scattered maximum or right-scattered minimum of $\mathbb{T}$ is called isolated.

Definition 2. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *right-dense continuous* or *rd-continuous* provided f is continuous at all right-dense points in $\mathbb{T}$, and $\lim_{s \to t} f(s)$ exists and is finite at all left-dense points in $\mathbb{T}$. The set of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$C_{\text{rd}} = C_{\text{rd}}(\mathbb{T}) = C_{\text{rd}}(\mathbb{T}, \mathbb{R}).$$

The set of functions that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{\text{rd}} = C^1_{\text{rd}}(\mathbb{T}) = C^1_{\text{rd}}(\mathbb{T}, \mathbb{R}).$$

We say f is *left-dense continuous* or *ld-continuous* provided f is continuous at all left-dense points in $\mathbb{T}$, and $\lim_{s \to t} f(s)$ exists and is finite at all right-dense points in $\mathbb{T}$.

Definition 3. The *graininess function,* $\mu : \mathbb{T} \to \mathbb{T}$, is defined by

$$\mu(t) := \sigma(t) - t.$$

The *backward graininess function,* $\nu : \mathbb{T} \to \mathbb{T}$, is defined by

$$\nu(t) := t - \rho(t).$$

Definition 4. The set $\mathbb{T}^\kappa$ is defined as follows. If $\mathbb{T}$ has a left-scattered maximum, M, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{M\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. Similarly, if $\mathbb{T}$ has a right-scattered minimum, m, then $\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}_\kappa = \mathbb{T}$.

1.2 Differentiation

Now we give some basic definitions regarding differentiation.

Definition 5. Assume $f : \mathbb{T} \to \mathbb{R}$, and let $t \in \mathbb{T}^\kappa$. Then we define the *delta-derivative of f at t*, denoted $f^\Delta(t)$, to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t, such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If $f^\Delta(t)$ exists we say f is *delta-differentiable*, and we call $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ the *delta-derivative* of f on $\mathbb{T}^\kappa$.

The nabla-derivative is defined in similar fashion:

Definition 6. Assume $f : \mathbb{T} \to \mathbb{R}$, and let $t \in \mathbb{T}_\kappa$. Then we define the *nabla-derivative of f at t*, denoted $f^\nabla(t)$, to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t, such that

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]| \leq \varepsilon|\rho(t) - s|$$

for all $s \in U$.

If $f^\nabla(t)$ exists for all $t \in \mathbb{T}$ we say f is *nabla-differentiable*, and we call $f^\nabla : \mathbb{T}_\kappa \to \mathbb{R}$ the *nabla-derivative* of f on $\mathbb{T}_\kappa$. This brings us the following theorems (from [4]) concerning the delta derivative:

Theorem 1. *(Theorem 1.16 [4]) Assume* $f : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}^\kappa$*. Then we have the following:*

(i) If f *is differentiable at* t*, then* f *is continuous at* t*.*

(ii) If f *is continuous at* t *and* t *is right-scattered, then* f *is differentiable at* t *with*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t *is right-dense, then* f *is differentiable at* t *if and only if the limit*

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists and is finite. In this case,

$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) If f *is differentiable at* t*, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t).$$

Theorem 2. *(Theorem 1.20 [4]) Assume* f, g *are differentiable at* $t \in \mathbb{T}^\kappa$*. Then:*

(i) The sum $f + g : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(f + g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$

(ii) For any constant α*,* $\alpha f : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$

(iii) The product $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* t *with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

(iv) If $f(t)f(\sigma(t)) \neq 0$*, then* $\frac{1}{f}$ *is differentiable at* t *with*

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

(v) If $g(t)g(\sigma(t)) \neq 0$*, then* $\frac{f}{g}$ *is differentiable at* t *with*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - g^{\Delta}(t)f(t)}{g(t)g(\sigma(t))}.$$

Definition 7. For a function $f : \mathbb{T} \to \mathbb{R}$, its second derivative $f^{\Delta\Delta}$ exists provided f^{Δ} is differentiable on $\mathbb{T}^{\kappa^2} := (\mathbb{T}^{\kappa})^{\kappa}$ and in this case $f^{\Delta\Delta} = (f^{\Delta})^{\Delta}$.

1.3 Integration

First, we need some preliminary definitions.

Definition 8. We say a function $f : \mathbb{T} \to \mathbb{R}$ is *regulated* provided its right-sided limits exist and are finite at all right-dense points in $\mathbb{T}$ and its left-sided limits exist and are finite at all left-dense points in $\mathbb{T}$.

Note that every rd-continuous function and every ld-continuous function is regulated.

Definition 9. A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (region of differentiation) $D \subseteq \mathbb{T}^{\kappa}$, provided $\mathbb{T}^{\kappa} \setminus D$ is a countable set containing no right-scattered element of $\mathbb{T}$, and f is differentiable at each $t \in D$.

Now we give some basic definitions regarding integration.

Theorem 3. *(Existence of Pre-Antiderivatives)(Theorem 1.70 [4]) Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^{\Delta}(t) = f(t) \qquad \textit{holds for all} \qquad t \in D.$$

Definition 10. Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Theorem 3 is called a *pre-antiderivative* of f. We define the *indefinite integral* of a regulated function f by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the *Cauchy integral* by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \qquad \text{for all} \qquad r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an (Δ-)*antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \qquad \text{holds for all} \qquad t \in \mathbb{T}^{\kappa}.$$

The following theorem states the existence of antiderivatives.

Theorem 4. *(Existence of Antiderivatives)(Theorem 1.74 [4]) Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau \qquad \textit{for} \qquad t \in \mathbb{T}$$

is an antiderivative of f.

Now we give some facts about integrals on time scales.

Theorem 5. *(Theorem 1.75 [4]) If $f \in C_{rd}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Theorem 6. *(Theorem 1.77 [4]) If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

(v) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$;

(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$;

(vii) $\int_a^a f(t)\Delta t = 0$;

(viii) *if* $|f(t)| \leq g(t)$ *on* $[a, b)$, *then*

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t;$$

(ix) *if* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\Delta t \geq 0$.

Parts (v) and (vi) in Theorem 6 are called integration by parts formulas.

Definition 11. If $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$, then we define the *improper integral of the first kind* by

$$\int_a^{\infty} f(t)\Delta t := \lim_{b\to\infty} \int_a^b f(t)\Delta t \quad b \in \mathbb{T}$$

provided this limit exists, and we say that the improper integral converges in this case. If this limit does not exist, we say the improper integral diverges.

1.4 Chain Rules

The chain rule as we know it from calculus does not hold in time scales calculus. For a simple example, let $f(t) = t^2$ and $g(t) = 2t$, where $\mathbb{T} = \mathbb{Z}$. We claim that

$$(f \circ g)^{\Delta}(t) \neq f^{\Delta}(g(t))g^{\Delta}(t).$$

Note that

$$f^{\Delta}(t) = 2t + 1 \quad \text{ and } \quad g^{\Delta}(t) = 2.$$

Also, $(f \circ g)(t) = 4t^2$, so

$$(f \circ g)^{\Delta}(t) = 8t + 4.$$

But,

$$f^{\Delta}(g(t))g^{\Delta}(t) = 8t + 2.$$

In time scales, there are several useful chain rules for differentiating composite functions. We will present two of them here.

Theorem 7. *(Chain Rule)(Theorem 1.87 [4]) Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^{\kappa}$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with*

$$(f \circ g)^{\Delta}(t) = f'(g(c))g^{\Delta}(t).$$

Notice that for $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, so $[t, \sigma(t)] = \{t\}$ and this gives the normal chain rule

$$(f \circ g)'(t) = f'(g(t))g'(t).$$

Theorem 8. *(Pötzsche Chain Rule)(Theorem 1.90 [4]) Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable, and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is*

delta differentiable and

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t).$$

Note that if $\mathbb{T} = \mathbb{R}$, $\mu(t) = 0$, so again as a special case, we get

$$(f \circ g)'(t) = \left\{ \int_0^1 f'(g(t))dh \right\} g'(t) = f'(g(t))g'(t).$$

1.5 Hilger's Complex Plane

In this section, we wish to develop the exponential function for time scales. In order to do this we need to discuss Hilger's complex plane. Many of the results from this section can be found in Hilger [19] and Bohner and Peterson [4].

Definition 12. For $h > 0$ we define the *Hilger complex numbers*, the *Hilger real axis*, the *Hilger alternating axis*, and the *Hilger imaginary circle* as

$$\begin{aligned} \mathbb{C}_h &:= \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}, \\ \mathbb{R}_h &:= \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z > -\frac{1}{h} \right\}, \\ \mathbb{A}_h &:= \left\{ z \in \mathbb{C}_h : z \in \mathbb{R} \text{ and } z < -\frac{1}{h} \right\}, \\ \mathbb{I}_h &:= \left\{ z \in \mathbb{C}_h : \left| z + \frac{1}{h} \right| = \frac{1}{h} \right\}, \end{aligned}$$

respectively. For $h = 0$, let $\mathbb{C}_0 = \mathbb{C}$, $\mathbb{R}_0 = \mathbb{R}$, $\mathbb{I}_0 = i\mathbb{R}$, and $\mathbb{A}_0 = \emptyset$.

Definition 13. Let $h > 0$ and $z \in \mathbb{C}_h$. We define the *Hilger real part* of z by

$$\mathrm{Re}_h(z) := \frac{|zh + 1| - 1}{h}$$

and the *Hilger imaginary part* of z by

$$\mathrm{Im}_h(z) := \frac{\mathrm{Arg}(zh+1)}{h},$$

where $\mathrm{Arg}(z)$ denotes the principal argument of z (i.e. $-\pi < \mathrm{Arg}(z) \leq \pi$).

Definition 14. We define *"circle plus"* addition $\oplus$ on $\mathbb{C}_h$ by

$$z \oplus w := z + w + zwh.$$

Lemma 1. *(Theorem 2.7 [4])* $(\mathbb{C}_h, \oplus)$ *is an Abelian group.*

Definition 15. We define *"circle minus"* subtraction $\ominus$ on $\mathbb{C}_h$ by

$$z \ominus w := z \oplus (\ominus w),$$

where $\ominus w$ is the additive inverse of w given by

$$\ominus w := -\frac{w}{1+wh}.$$

Definition 16. If $n \in \mathbb{N}$ and $z \in \mathbb{C}_h$, then we define *"circle dot"* multiplication $\odot$ by

$$n \odot z := z \oplus z \oplus z \oplus \cdots \oplus z.$$

where we have n terms on the right-hand side of this equation.

Definition 17. For $h > 0$ let $\mathbb{Z}_h$ be the strip

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \mathrm{Im}(z) \leq \frac{\pi}{h} \right\},$$

and for $h = 0$, let $\mathbb{Z}_0 := \mathbb{C}$

Definition 18. For $h > 0$, we define the *cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) = \frac{1}{h}\mathrm{Log}(1 + zh),$$

where Log is the principal logarithm function. For $h = 0$, define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

1.6 The Exponential Function

In this section we will use the cylinder transformation to define a generalized exponential function for an arbitrary time scale $\mathbb{T}$. First we need some preliminary definitions.

Definition 19. We say that the function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^\kappa$$

holds. The set of all regressive and rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

A function p is positively regressive if it is regressive and

$$1 + \mu(t)p(t) > 0 \quad \text{for all} \quad t \in \mathbb{T}^\kappa.$$

The set of positively regressive functions will be denoted by

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}) = \mathcal{R}^+(\mathbb{T}, \mathbb{R}).$$

Definition 20. Define *"circle plus"* addition $\oplus$ on $\mathcal{R}$ by

$$(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t) \quad t \in \mathbb{T}$$

and *"circle minus"* subtraction $\ominus$ by

$$(p \ominus q)(t) := p(t) \oplus (\ominus q(t)) \quad t \in \mathbb{T},$$

where

$$\ominus q(t) = -\frac{q(t)}{1 + \mu(t)q(t)}.$$

Theorem 9. *The set $(\mathcal{R}, \oplus)$ is an Abelian group.*

Proof. Notice that the zero function $0 \in \mathcal{R}$ and for all $p \in \mathcal{R}$, we have $(p \oplus 0)(t) = p(t)$. Thus, 0 is the identity under circle-plus addition. Also,

$$\begin{aligned} 1 + \mu(t)(\ominus p)(t) &= 1 + \frac{-\mu(t)p(t)}{1 + \mu(t)p(t)} \\ &= \frac{1 + \mu(t)p(t) - \mu(t)p(t)}{1 + \mu(t)p(t)} \\ &= \frac{1}{1 + \mu(t)p(t)} \neq 0. \end{aligned}$$

So we have $\ominus p \in \mathcal{R}$ for all $p \in \mathcal{R}$. Also,

$$\begin{aligned} (p \oplus (\ominus p))(t) &= p(t) + \frac{-p(t)}{1 + \mu(t)p(t)} + \mu(t)p(t)\frac{-p(t)}{1 + \mu(t)p(t)} \\ &= \frac{p(t) + \mu(t)p^2(t) - p(t) - \mu(t)p^2(t)}{1 + \mu(t)p(t)} \\ &= 0. \end{aligned}$$

Therefore, $\ominus p$ is the inverse of p for all $p \in \mathcal{R}$.

Next, we have that $\mathcal{R}$ is closed under $\oplus$: Let $p, q \in \mathcal{R}$. Then

$$\begin{aligned} 1 + \mu(t)(p \oplus q)(t) &= 1 + \mu(t)p(t) + \mu(t)q(t) + \mu^2(t)p(t)q(t) \\ &= (1 + \mu(t)p(t))(1 + \mu(t)q(t)) \neq 0. \end{aligned}$$

Finally, if we let $p, q, r \in \mathcal{R}$, then we have

$$\begin{aligned} (p \oplus (q \oplus r))(t) &= p(t) + (q \oplus r)(t) + \mu(t)p(t)(q \oplus r)(t) \\ &= p(t) + q(t) + r(t) + \mu(t)q(t)r(t) \\ &\quad + \mu(t)p(t)(q(t) + r(t) + \mu(t)q(t)r(t)) \\ &= p(t) + q(t) + r(t) + \mu(t)q(t)r(t) \\ &\quad + \mu(t)p(t)q(t) + \mu(t)p(t)r(t) + \mu^2(t)p(t)q(t)r(t)) \\ &= p(t) + q(t) + \mu(t)p(t)q(t) + r(t) \\ &\quad + \mu(t)r(t)(p(t) + q(t) + \mu(t)p(t)q(t)) \\ &= (p \oplus q)(t) + r(t) + \mu(t)(p \oplus q)(t)r(t) \\ &= ((p \oplus q) \oplus r)(t). \end{aligned}$$

Therefore, $(\mathcal{R}, \oplus)$ is an Abelian group. $\square$

Definition 21. If $p \in \mathcal{R}$ then we define the *exponential function* by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \qquad \text{for} \qquad s, t \in \mathbb{T},$$

where the cylinder transformation $\xi_h(z)$ is as in Definition 18.

Lemma 2. *(Lemma 2.31 [4]) If $p \in \mathcal{R}$ then the following semigroup property holds:*

$$e_p(t,r)e_p(r,s) = e_p(t,s) \qquad \text{for all } r, s, t \in \mathbb{T}.$$