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BOUNDARY VALUE PROBLEMS FOR NONLINEAR
DIFFERENTIAL EQUATIONS.

The University of Nebraska - Lincoln, Ph.D., 1974
Mathematics

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BOUNDARY VALUE PROBLEMS FOR NONLINEAR DIFFERENTIAL EQUATIONS

by

Dwight V. Sukup

A DISSERTATION

Presented to the Faculty of

The Graduate College in the University of Nebraska

In Partial Fulfillment of Requirements

For the Degree of Doctor of Philosophy

Department of Mathematics

Under the Supervision of Professor Allan C. Peterson

Lincoln, Nebraska

June, 1974

TITLE

Boundary Value Problems for

Nonlinear Differential Equations

BY

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Boundary Value Problems for

Nonlinear Differential Equations

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University of Nebraska, 1974

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In this work we consider the differential equation

$$(1) \qquad y^{(n)} = f(x, y, y', \ldots, y^{(n-1)})$$

where we assume that f is continuous on $[\alpha, \beta) \times R^n$ and that solutions of initial value problems are unique and extend throughout $[\alpha, \beta)$. In some cases we will assume that solutions of the equation (1) satisfy a certain compactness condition and in some cases we will assume local uniqueness of solutions of boundary value problems for the equation (1).

We begin by defining the boundary value sets $S(y(x); x_1^{i_1}, \ldots, \hat{x}_k^{i_k}, \ldots, x_m^{i_m})$. We show that these boundary vale sets satisfy certain set theoretic properties and subsequently use these boundary value sets to obtain several uniqueness implies existence results for boundary value problems for the equation (1).

Finally, we consider extremal solutions for the nonlinear equation (1). In particular, we are able to obtain some

generalizations of a result due to T. L. Sherman to the nonlinear case.

Acknowledgement

The author wishes to express his sincere thanks to Professor Allan C. Peterson for his help, understanding, and encouragement during the preparation of this dissertation.

CONTENTS

1. Introduction.

We will consider the following nonlinear differential equation in this work:

$$y^{(n)} = f(x, y, y', \ldots, y^{(n-1)}) \tag{1.1}$$

where $x \in I = [\alpha, \beta)$, $\alpha < \beta \leq +\infty$.

The following assumptions are assumed to be satisfied throughout this work:

(A) f is continuous on $[\alpha, \beta) \times R^n$, and

(B) solutions of initial value problems (IVP's) are unique and extend throughout $[\alpha, \beta)$.

<u>Definition 1.1</u>. We say that $y \in C^n[\alpha, \beta)$ has an $(i_1, \ldots, i_m)$-distribution of zeros, $0 \leq i_k \leq n$, $\sum_{k=1}^{m} i_k = n$, on $[c, d] \subset [\alpha, \beta)$ provided there are points $c \leq x_1 < \ldots < x_m \leq d$ such that $y(x)$ has a zero of order at least i_k at x_k, $k = 1, \ldots, m$.

<u>Definition 1.2</u>. Let $R = \{ r > t:$ there exist distinct solutions $u(x)$ and $v(x)$ of equation (1.1) such that $u(x) - v(x)$ has an $(i_1, \ldots, i_m)$-distribution of zeros, $0 \leq i_k \leq n$, $\sum_{k=1}^{m} i_k = n$, on $[t, r]\}$. If $R \neq \phi$ set $r_{i_1 \ldots i_m}(t) = \inf R$. If $R = \phi$, set $r_{i_1 \ldots i_m}(t) = +\infty$.

Remark 1.3. If $t \le x_1 < \cdots < x_m < r_{i_1 \ldots i_m}(t) \le +\infty$, then solutions of equation (1.1) satisfying the boundary conditions

$$(1.2) \qquad y^{(l_j - 1)}(x_j) = C_{j,l_j} \;;\; C_{j,l_j} \in R^1 = (-\infty, \infty);$$
$$l_j = 1,\ldots,i_j \;;\; j = 1,\ldots,m$$

when they exist are unique.

Equation (1.1) along with boundary conditions (1.2) is called an $(i_1,\ldots,i_m)$-boundary value problem (BVP) and will be referred to in this paper as the $(i_1,\ldots,i_m)$-BVP (1.1), (1.2). In the linear case it is well known that if $t \le x_1 < \cdots < x_m < r_{i_1 \ldots i_m}(t) \le +\infty$, then for the given linear equation there always exists a unique solution satisfying the boundary conditions (1.2). Uniqueness implies existence results of this type for the nonlinear case is the main concern of Chapter 2 of this work.

Definition 1.4. The first conjugate point $\eta_1(t)$ for the nonlinear equation (1.1) is defined by

$$\eta_1(t) = \min\{r_{i_1 \ldots i_m}(t) : \sum_{k=1}^{m} i_k = n\}$$

Definition 1.5. If I is an interval and $I \subset [\alpha, \beta)$ then we say that I is an interval of disconjugacy for equation (1.1) provided there do not exist distinct solutions $y(x)$, $z(x)$ of equation (1.1) such that $y(x) - z(x)$ has at least n zeros, counting multiplicities, on I. It is clear from Definition 1.4

that if $I \subset [t, \eta_1(t))$ then I is an interval of disconjugacy and that $[t, \eta_1(t))$ is a maximal half open interval of disconjugacy.

Remark 1.6. It is possible that $\eta_1(t) = t$ for the nonlinear equation (1.1). For example, if $n = 2$ and we consider the equation

$$y'' = -y^3$$

it was shown in [11] that for this equation $\eta_1(t) = t$ for all t. However, if f satisfies a uniform Lipschitz condition with respect to $y, y', \ldots, y^{(n-1)}$ on compact subintervals of $[\alpha, \beta)$, then, using estimates for the Green's function 4 and the fact that $\eta_1(t) = r_{1\ldots1}(t)$, [3], one can use standard fixed point arguments to prove that $\eta_1(t) > t$ for all $t \in [\alpha, \beta)$. In particular, if equation (1.1) is a linear differential equation, then, as is well known, $\eta_1(t) > t$ for all t. Some of the results of Chapter 2 and Chapter 3 of this work require the hypothesis that $\eta_1(t) > t$ for some $t \in [\alpha, \beta)$. Later in this introduction estimates for bounds on the Green's function for the $(n-1,1)$-BVP (1.1), (1.2) and standard fixed point arguments will be used to prove that $r_{n-1,1}(t) > t$ for all $t \in [\alpha, \beta)$ in a special case. In particular we will assume that f satisfies a uniform Lipschitz condition with respect to $y, y', \ldots, y^{(n-1)}$ on each compact subinterval of $[\alpha, \beta)$ and establish lower bounds for $r_{n-1,1}(t)$.

<u>Definition 1.7</u>. Let $y(x)$ be a solution of equation (1.1) and let $\alpha \le t < x_1 \quad \ldots < x_m < \beta$. Define

$$S(y(x);\, x_1^{i_1}, \ldots, \hat{x}_k^{i_k}, \ldots, x_m^{i_m}) \equiv$$

$\{ u^{(i_k-1)}(x_k) :$ $u(x)$ is a solution of equation (1.1) such that

$$u^{(l_j)}(x_j) = y^{(l_j)}(x_j)\ ,\ l_j = 0, \ldots, i_j-1; \quad j = 1, \ldots, m; \quad j \neq k$$

$$u^{(l_k)}(x_k) = y^{(l_k)}(x_k)\ ,\ l_k = 0, \ldots, i_k-2$$

(if $i_k = 1$ there is no boundary condition at x_k)$\}$.

If the superscript i_j is 1 then the superscript will be omitted and understood to be 1.

Let $\alpha \le x_1 < x_2 < \beta$ and let $y(x)$ be a solution of

(1.3) $$y'' = f(x, y, y') \ .$$

The set $S(y(x);\, x_1, \hat{x}_2)$ is a connected set. To see this let $\phi(\lambda) = u_\lambda(x_2)$ where $u_\lambda(x)$ is the solution of equation (1.3) such that $u_\lambda(x_1) = y(x_1)$ and $u_\lambda'(x_1) = \lambda$. It follows by condition (B) that $\phi(\lambda)$ is a continuous function of λ. Therefore $S(y(x);\, x_1, \hat{x}_2) = \phi(R^1)$ must be a connected set. The following example shows that by the proper selection of $f(x, y, y')$ for equation (1.3) we can determine $S(y(x);\, x_1, \hat{x}_2)$ to be any open, half open or closed interval.

Example 1.8. Let $a < b$ and define $f(x, y, y')$ by

$$f(x, y, y') = \begin{cases} -y + b & , y \geq b \\ 0 & , (a + b)/2 < y < b \\ -y + (a + b)/2 & , y \leq (a + b)/2 \end{cases}$$

For the solution $y(x) \equiv b$ of equation (1.3) we have that $S(y(x); 0, \hat{\pi}) = (a, b]$. To see this we let $u_\lambda(x)$ be the solution of equation (1.3) such that $u_\lambda(0) = b$ and $u_\lambda'(0) = \lambda$. If $\lambda > 0$ then $u_\lambda(x) = b + \lambda \sin x$ and $u_\lambda(\pi) = b$. If $\lambda \leq 0$ and $b + \lambda\pi \geq (a + b)/2$ ($(a - b)/2\pi \leq \lambda \leq 0$) then $u_\lambda(x) = b + \lambda x$ for $x \in [0, \pi]$ and $u_\lambda(\pi) = b + \lambda\pi$. Hence $[(a + b)/2, b] \subset S(y(x); 0, \hat{\pi})$. If $b + \lambda\pi < (a + b)/2$ ($\lambda < (a - b)/2\pi$) then for $x \in [0, \pi]$

$$u_\lambda(x) = \begin{cases} b + \lambda x & , 0 \leq x \leq (a-b)/2\lambda \\ (a+b)/2 + \lambda \sin(x - (a-b)/2\lambda) & , (a-b)/2\lambda < x \leq \pi \end{cases}$$

and hence

$$\begin{aligned} u_\lambda(\pi) &= (a+b)/2 + \lambda \sin(\pi - (a-b)/2\lambda) \\ &= (a+b)/2 + ((a-b)/2)(2\lambda/(a-b))\sin((a-b)/2\lambda). \end{aligned}$$

It follows that $u_\lambda(\pi)$ is a continuous monotonically increasing function of λ for $\lambda < (a-b)/2\pi$ and that

$$\lim_{\lambda \to -\infty} u_\lambda(\pi) = (a+b)/2 + (a-b)/2 = a \quad .$$

This argument has established that for the solution $y(x) \equiv b$ of equation (1.3) we have $S(y(x); 0, \hat{\pi}) = (a, b]$.

If $f(x, y, y')$ is defined by

$$f(x, y, y') = \begin{cases} -y + (a+b)/2 & , y \geq (a+b)/2 \\ 0 & , a < y < (a+b)/2 \\ -y + a & , y \leq a \end{cases}$$

then in an analogous way we can show that for the solution $y(x) \equiv a$ of equation (1.3) we have $S(y(x); 0, \hat{\pi}) = [a, b)$.

If $f(x, y, y')$ is defined by

$$f(x, y, y') = \begin{cases} -y + (a+3b)/4 & , y \geq (a+3b)/4 \\ 0 & , (3a+b)/4 < y < (a+3b)/4 \\ -y + (3a+b)/4 & , y \leq (3a+b)/4 \end{cases}$$

then for the solution $y(x) \equiv (a+b)/2$ of equation (1.3) we have that $S(y(x); 0, \hat{\pi}) = (a, b)$.

If $f(x, y, y')$ is defined by

$$f(x, y, y') = \begin{cases} -y + (b/\pi)x & , y \geq (b/\pi)x \\ 0 & , (a/\pi)x < y < (b/\pi)x \\ -y + (a/\pi)x & , y \leq (a/\pi)x \end{cases}$$

then for the solution $y(x) = (a/\pi)x$ of equation (1.3) we have that $S(y(x); 0, \hat{\pi}) = [a, b]$. To see this we let $u_\lambda(x)$ be the solution of equation (1.3) such that $u_\lambda(o) = 0$ and $u_\lambda'(0) = \lambda$. If $a/\pi \leq \lambda \leq b/\pi$ we have that $u_\lambda(x) = \lambda x$ and $u_\lambda(\pi) = \lambda\pi$. If $\lambda > b/\pi$ we have that $u_\lambda(x) =$