COGNITIVE CONSTRUCTS IN LINEAR ALGEBRA; METAPHORS, METONYMIES, MODES

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Abstract

Analysis focused on the presence of different thinking modes, metonymies, and metaphors found on the interview responses to questions related to linear independence, span, and spanning sets of three students, A12, A22 and C3, taking their first linear algebra course at the college level. Findings provide insight into how first year linear algebra students move from one thinking mode to another, and the kind of metonymies and metaphors are used to construct new knowledge. The main purpose of this research was to discover and analyze the presence of the different modes of thinking and metonymy/metaphors– in the reasoning of these three students.

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Chapter 1

Introduction

The career of students at the university level requires them to take many courses that are crucial for their field of study and for the future expansion of related topics. Many times students do not realize the difficulty level of concepts and what is required to master them. In fact matrix algebra is among those courses students tend to overlook its cognitive difficulties and its role in their subsequent studies. Linear algebra, a course that has an immense range of applications in different disciplines - is a requirement in the curricula of many subjects such as mathematics, computer science, and engineering majors.

The vast range of use of linear algebra ideas requires a close attention to the subject in the training of university level students. Students often find themselves struggling to understand, explain, and relate the theory learned while enrolled in their first linear algebra course, because of the different kinds of representations employed. It has been discovered that students do not have enough previous knowledge in mathematical structures such as algebraic skills and set theory (Dogan-Dunlap, 2006; 2010) to construct new concepts and relate them to previously learned materials (Sierpinska, 2000). This becomes a discouragement and a problem for students resulting in cutting short of the learning process or discourage from fully grasping the new material being presented.

This paper is part of an ongoing research funded by the National Science Foundation (NSF; CCLI:0737485) that focuses on the teaching methods used in first year linear algebra courses at university level, the conceptual constructs that students display while enrolled in them, and the importance that different visual representations have in the development of knowledge.

The purpose of the thesis is to reveal the different thinking modes and the examples of metonymy and metaphors exhibited by three students on their responses to a set of questions asked during one-on-one interviews conducted toward the end of their first matrix algebra course. Our goal is to further document the understanding and misconceptions that students display in light of different levels of exposure to graphical, computational, algebraic, and abstract representations of basic topics such as linear independence and dependence, span, spanning set, and vector spaces. Our goal will be achieved by addressing the following question: What are the thinking modes, metonymies and metaphors displayed by three students in their responses to interview tasks on linear independence?

1.1 LEARNING THEORIES

The study investigated the occurrence of cognitive entities— modes of thinking and metonymy/metaphors. Throughout the analysis, these cognitive entities will be recorded and materialize in the student responses.

1.1.1 Modes of Thinking

Anna Sierpinska's structure in her paper "On some aspects of students' thinking in linear algebra"(2000) will be used to document the modes of thinking displayed by the students involved in our research through the qualitative analysis of their responses in the interviews. Sierpinska documents three different thinking modes; Synthetic-Geometric, Analytic-Arithmetic, and Analytic-Structural employing the origins and the characteristics of linear algebra concepts and the mathematical languages used in learner's understanding.

1.1.1.1 Synthetic-Geometric

The use of geometric representations and the lack of definitions for the concepts used in objects is a case of synthetic-geometric modes; for example, in the form of a line or a plane the properties of objects such as linear independence and span can be given by student, but they will only describe such objects, not define them (Sierpinska, 2000). In other words, using geometry in such a way in which students think visually and use it as the basis for their knowledge.

1.1.1.2 Analytic-Arithmetic

Analytic-Arithmetic mode considers an object defined which is then used to carry out computations (Sierpinska, 2000). The numerical and algebraic components of geometrical objects and the conditions they satisfy are employed. Computing the numerical and algebraic components refer to carrying out appropriate algorithms, while analytically refers to reasoning and justifying facts and algorithms confidently.

1.1.1.3 Analytic-Structural

In the analytic-structural mode objects are defined by its properties and, although algebraic structures are still considered as in the analytic-arithmetic mode, in the structural mode they are synthesized into compact structural wholes (Sierpinska, 2000). Students thinking of vectors as being part of and having characteristics of vectors spaces and proving linear independence of a set of vectors through the use of its dimension arguments are examples of the implementation of this perspective (Dogan-Dunlap, 2008; 2010; 2011 a, b, c revise the references on Dogan adding coauthors if there are coauthors on citations; I have included a few more articles on references to be cited in the body of the paper especially lit chapter ????????).

Below is a summary of the different thinking modes identified by Sierpinska, examples of their representations, and the level of competency that a student may achieve when using each one of them is presented in table 1.

Mode of Thinking	**Representations/Definition**	**Student Competency**
Synthetic-Geometric	Graphical representations Provide properties of objects readily. It describes an object but not define it.	Student is be able determine whether vectors whose graphs are provided in R^2 or R^3 are linearly independent or dependent.
Analytic-Arithmetic	Numerical Representations. Defines objects.	Student is able to construct matrix from vectors, compute its row-reduced echelon form and relate the reduced matrix to linear dependence and independence.
	Linear Combination.	Student is able to provide/refer to linear combination of vectors and determine linear independence.
Analytic-Structural	Objects are considered in a system. Defines objects.	Use of the dimension of vector spaces in determining the linear independence of vectors.

Table 1. Thinking Modes Modified from Sierpinska (obtained from Dogan-Dunlap, 2010)

It is worth mentioning that a student can and will use different modes of thinking and shift between them in order to understand and explain concepts (Sierpinska, 2000).

1.1.2. Metonymy and Metaphor

The constructs, metonymy and metaphor, are widely used primarily as examples of figurative speech or literary devices. Recently however they also been considered as cognitive entities aiding one's learning process. Presmeg (1998) is among the researchers who began document their cognitive roles in learner's cognition.

1.1.2.1 Metaphor

Defined by Webster's dictionary a metaphor is "a figure of speech in which an expression denoting one kind of object is used in place of another in order to suggest a similarity" (Webster). The use of this literary device in the learning of mathematics is employed by students when acquiring new knowledge; its application involves the comparison of two entities. There are two very important characteristics of the comparison between the two entities: the ground and the tension (Presmeg, 1998). Presmeg clarifies the grounds as the similar elements of the entities being compared, while the tension as the dissimilar elements (1998).

An example on the use of metaphors in mathematics can be seen in the statement, A is an open set; here, the definition of openness and its physical representation of being without a boundary can become a source of confusion for students since they may wrongfully interpret the "no boundary characteristic of the source concept, and come to a conclusion that the particular set is not open since it has a boundary" (Dogan-Dunlap, 2007, page 2). This example illustrates the importance of the identification of the tension and ground elements in the comparison. An additional example of the use of metaphors in the understanding of mathematical concepts is given by Presmeg through the responses of high school students who were asked to calculate the sum of the first 30 elements of the sequence {5, 8, 11…} (1998). In such study, students referred to their methods by using metaphors such as "dome" and "rainbow" to refer to the Gaussian way

of having to relate the first and last elements of the sequence and adding them, then the second and the 29th, the third and the 28th, and so on to reach their final answer.

1.1.2.2 Metonymy

Lakoff and Johnson (2000) refer to metonymy as being present when one uses "one entity to refer to another that is related to it." While Webster's dictionary definition is "a figure by which one word is put for another on account of some actual relation between the things signified". Some examples commonly used to refer to this literary device are the use of the phrases "We read "Shakespeare" when we talk about the author's work" and "Washington is talking to Moscow" when we talk about the people from these countries communicating (Webster). The principal characteristic of this figure of speech is the presence of an attribute or entity that is taken to stand for another entity (Presmeg, 1998).

1.1.2.3 The relationship between metaphor and metonymy

Both metaphor and metonymy devices are closely related to the understanding and development of new knowledge form. These two tools are widely used by students with mathematical backgrounds in which a person uses one construct to stand for another for example students using their own language or terminology to understand and relate to the new concept.

The main difference between the two devices is that through the use of metaphors, the learner would make connections based on similarity in contrast to metonymy which are made by association with no regards to similarity. This relationship is often seen when a student recently introduced to the term inverse referring to the steps of finding the inverse of a function, the word "inverse" becomes part of the concept "additive inverse" performing operations that require applying the opposite sign of a number.

1.2 RESEARCH QUESTIONS

The purpose of this thesis is to identify cognitive constructs employed in concept formation by analyzing the responses given by three students from the different modular and traditional groups to a set of questions asked during interviews. This goal will be achieved by addressing the following questions:

- What are the thinking modes displayed by three students in their responses to interview tasks on linear independence?
- What are the metonymies and metaphors displayed by the three students in such responses?

1.3 METHODOLOGY

For the purpose of this thesis, we'll analyze the responses of three undergraduate students, enrolled in their first linear algebra course, to a set of eight questions asked during their one-on-one interviews scheduled toward the end of the Spring 2009 semester. Each of these students was selected at random from a list of volunteers interviewed at the end of the semester and belonging to three different courses. Our main goal is to analyze the different aspects of learning shown by each student through the presence of the distinct modes of reasoning and the use of metaphors/metonymy as part of their responses.

1.3.1 Participants

The students that participated in this research came from three different groups taking a first year linear algebra course during the Spring 2009 semester. Due to the demographic

location of the region where this study took place, the majority of the students from the groups were Hispanic and a considerable percentage had English as their second language.

Two of the groups were enrolled in what was referred to as a modular matrix algebra course (non-traditional course), while the other was called non-modular course (traditional course). The modular matrix algebra courses enforced the use of computerized mathematical modules, accessed through the internet, that were introduced as part of the class and related homework assignments. On the other hand, the non-modular course had a traditional approach, where the professor lectured and assigned homework, but the computerized modules were not included or even mentioned.

Since these classes were taught by different professors with different teaching techniques and hence, different levels of abstraction, it's important to state that the generalization of these results to the entire population of linear algebra students had to be made carefully, as there were important differences even within the two modular courses.

1.3.2 Modular and Non-modular Section Characteristics

In the modular versions of the course, topics were presented during class to students through a formal definition, row reduced echelon operations, algebraic manipulations, and, frequently through the graphical representations. Homework from the required textbook was often assigned (but not collected) and an assignment, to be answered through the use of the computer modules, was administered and collected (students had an average of a week to work on the material). Sometimes, professors would introduce new topics by using the computer modules through an overhead projector and would explain the characteristics of the new topic and the relationship to past topics.

In the non-modular version of the course, topics were presented during the class through formal definition, row reduced echelon operations and, depending on the questions asked by students, sometimes the professor would provide a graphical representation of the topic. Homework was assigned (but not collected) and consisted on problems taken mainly from the required textbook (homework question were given in quizzes in one of the modular groups).

The official course description for both, the modular and the non-modular sections of this class during the spring 2009 semester is as follows:

> *MATRIX ALGEBRA 3323: Systems of linear equations, matrices, determinants, eigenvalues and eigenvectors, diagonalization, vector spaces and linear transformations.*

However, the topics chosen for this thesis is limited to:

1. The definition of linear dependence or independence in a set of vectors; identification of linear dependency in particular sets of vectors.
2. Characteristics of linearly independent/ dependent set of vectors in R^2 and R^3.

1.4 ANALYSIS

A qualitative approach, namely the constant comparison method (Glaser, 1992), is used to analyze student responses on the interviews. The qualitative analysis focused on the presence and categorization of thinking modes and metaphors/metonymy in students' interview responses to questions about linear independence, span, and spanning sets.

1.4.1 Qualitative Analytical Procedures

The interviews of 3 students – one from each of the classes available during the Spring 2009 semester— were transcribed and summarized. The qualitative analyses of the transcripts were conducted by the author of this thesis, his advisor, and an additional graduate student with strong mathematics background and will consist in the identification and classification of the presence of the cognitive constructs defined previously: thinking modes and metaphors/metonymy. An inter-reliability test was conducted for each interview to rate the consensus between the raters. Discussion among the raters was done continuously to discuss the different categories identified in each interview and was completed when no additional categories emerge. Once all possible categories were listed, sample student responses and category descriptions included in each as identifiers. Afterwards, the frequency and types of thinking modes and metaphors/metonymy identified were recorded for each student in order to address our research questions.

Chapter 2

Literature Review

Research to discover the various thinking modes students display from linear algebra, has displayed to be somewhat cumbersome and limited. The teaching community believes that courses to teach linear algebra are not being presented in a format comprehendible to students. There are others who believe linear algebra will always be challenging for students no matter how it is presented (Dorier et al., 2001).

It was discovered that many students possess learning difficulties with basic linear algebra concepts (Dogan-Dunlap, 2010). Documented for several years these difficulties have help gear changes to increase the level of understanding of linear algebra students at university level. Despite these changes, radical outcomes have not been apparent because of the variety of thinking modes students display, and difficulties that still plague students today.

2.1 Epistemological Aspects of Linear Algebra

The challenge to learn mathematics for many students is by memorization. They soon discover as the level of difficulty increases this does not work. This style of learning offers a basic knowledge and may aide a student to succeed in a class, but presents future hurdles in upper level math. Memorization for many college students has been their underlying method of learning throughout their post college carrier making it challenging to create a link to their mathematical college level courses. Linear algebra is one of those courses, with a high level of abstraction; gives the presence to be particularly problematic to most students. So much so that students become discouraged not being able to comprehend the material, as a result, their familiarity structures become disjointed with absence reasoning. Dogan-Dunlap (2010) indicate

that some of the difficulties students face in linear algebra courses are the ―high level of formalism" and the "axiomatic approach" for which students are not equipped to grasp.

2.1.1 The Conceptual Aspects of Linear Algebra

Dorier and Sierpienska (2001) believe there is a "necessity of cognitive flexibility" for an insightful understanding of linear algebra concepts. Many students have trouble connecting different visual representations used to represent linear algebra concepts due to their lack of logic and set theory knowledge (Dogan-Dunlap, 2006). As mentioned in Dogan-Dunlap (2010, pp. 2), Dubisky and Harrel (1997) explained that students are capable of achieving abstraction if the flexibility between the representations of the same concept is instituted. Abstraction is established if concept images and concept definitions are not contradicting each other (Dogan-Dunlap, 2010).

After 1930, a theoretical reconstruction of the methods to solve linear algebra problems initiated a new axiomatic central theory (Dorier et al, 2001). Dorier and Sierpinska (2001) state the new axiomatic central theory gave linear algebra a more universal approach and language to be used in different contexts. This new theory also involved the use of concepts and tools that were not explicitly formulated or unified, and it marked a new level in abstraction (Dorier et al, 2001). The different perspective brought a more sophisticated level in mental operations that as a result, later manifested in difficulties associated with the pre-existing related elements of knowledge from lower levels (Dorier et al, 2001).

To have a solid understanding of linear algebra concepts, students need to first 'concretize' these abstract objects and their representations (Dorier et al, 2001). Most linear algebra students are overwhelmed by the amount of new definitions and theorems and with the

high level of formalism students seem to have a lack of connection to what they already know (Dogan-Dunlap, 2006, 2010).

According to Hillel (2000) linear algebra can be represented with the use of three basic languages, such as geometric, algebraic, and abstract. The geometric language of two and three dimensional spaces includes line segments, points, geometric transformations, and planes. The algebraic language of the R^n space, includes n-tuples, matrices, and rank and the abstract language states to abstract theory, such as vector spaces, linear transformations of vector spaces, and the eigen value theory (Dorier et al, 2001). Hillel (2000) discovered that the way instructors used to shift from one language to the other, without any pause or attempt to alert students of the change, deprived the students of the time needed to assimilate the relationships among the concepts being learned.

2.1.2 The Cognitive Characteristics of Linear Algebra

Semiotic representations, as defined by Duval (1995) are "productions made by the use of signs belonging to a system of representation which has its own constraints of meaning and functioning". Duval states these representations are 'absolutely necessary' in mathematics because some objects cannot be directly recognized and must be represented (Dorier et al, 2001). Semiotic representations play an important role in the development of mental representations, accomplishment of cognitive functions, and production of knowledge (Dorier et al, 2001). According to Duval (1995) semiosis and noesis -the highest cognitive process- are two acts that cannot be separated from each other, but they differ in that the first refers to "the comprehension or production of a representation by a sign" while the second refers to "the conceptual comprehension of an object". Duval identified three types of cognitive activities related to

semiosis, the formation of a representation, the processing and transformation of a representation, and the conversion of a semiotic representation from one register to another (Dorier et al, 2001).

Pavlopoulou (see Dorier 2000, pp. 247-252) was able to distinguish between three registers of semiotic representation of vectors; arrows as the graphical register, columns of coordinates as the table register, and finally the axiomatic theory of vector spaces as the symbolic register. Pavlopoulou on her research also discovered confusion among the students with respect to an object and its representation and difficulty in converting from one register to another (Dorier et al, 2001). As described by Dorier and Sierpinska (2001), Alves-Dias was able to “generalize the necessity of conversions from one semiotic register to another for the understanding of linear algebra to the necessity of cognitive flexibility”. Registers of semiotic representation requires the student to be able to move from one to another (Dorier et al, 2001).

Another cognitive requirement of linear algebra students is the need for background knowledge in areas such as, set theory, logic, and proofs (Zamora, 2010). According to Dogan-Dunlap (2006), Bogomolny (2007), and Rogalski (2000) some of the problems that linear algebra students face manifest due to the lack of background knowledge in the those areas.

2.2 Principles of Teaching Linear Algebra

The United States established in 1990 a movement to develop the learning and teaching of linear algebra named the Linear Algebra Curriculum Study Group (LACSG) to report concerns of the teaching and learning of linear algebra. The LACSG, composed by sixteen mathematics educators from across the country, created a list of recommendations based on a combination of three major sources, research-based knowledge done on students‘ learning

processes and the optimal teaching methods of linear algebra, individual teaching experience of LACSG members, and the contribution of consultants from various disciplines who explained how linear algebra was related to their field and what kind of changes in the curriculum could benefit them (Harel, 2000).

The LACSG members made five major recommendations to improve the teaching and learning of linear algebra (Harel, 1997).

- The first course in linear algebra should not be entirely focused on proofs
- A second course of linear algebra should be part of every mathematics curriculum
- The incorporation of technology
- The introduction of linear algebra concepts in high school
- A core syllabus that included concepts such as matrix addition and multiplication, Gaussian elimination, echelon and reduced echelon form, matrix inverses, determinants, linear combinations, linear dependence and independence, subspaces of R^n , bases of R^n, matrices as linear transformations, rank, inner products, eigen vectors, eigen values, in between others (Harel, 2000).

Following these recommendations Harel (2000) developed a theoretical framework based on the three learning-teaching principles: the Concreteness Principle, the Necessity Principle, and the Generalizibility Principle.

2.2.1 Concreteness Principle

After working on experiments with high school and beginning college students, Harrel (2000) found the assumption of students being able to deal with abstract structures without