ELASTIC THERMAL STRESSES IN THICK CYLINDRICAL
AND SPHERICAL SHELLS WITH
INTERNAL HEAT GENERATION

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Dean of the Graduate School

In memory of my father

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THESIS

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ABSTRACT

The elastic thermal stresses in thick cylindrical and spherical shells with internal heat generation have been studied analytically. Dimensionless stress functions have been developed. The computer generated graphs for the stress functions for various outside to inside radii ratios ($1.25 < R_o = r_o/r_i, < 3.0$) are presented to facilitate computations of the stresses at any radial location of the shell.

In order to make comparisons with the solid cylindrical and solid spherical cases with internal heat generation, the stresses functions for them, have also been developed and presented graphically.

The location for the maximum value of the dimensionless stress functions for appropriate cases has been determined and shown on the graphs.

CONTENTS

LIST OF FIGURES

NOMENCLATURE

C_1, C_2	Constants of integration
C_p	Specific heat; $Btu/lb_m \cdot °F$
E	Modulus of elasticity; lb_f/ft^2
F_h, F_r, F_z	Dimensionless stress functions
H	Uniform heating density; $Btu/hr \cdot ft^3$
k	Thermal conductivity; $Btu/hr \cdot ft \cdot °F$
p	Hydrostatic pressure; lb_f/ft^2
r	Local radius; ft
r_i	Inner radius; ft
r_o	Outer radius; ft
R	Radius ratio; r/r_i
R_o	Radius ratio; r_o/r_i
T	Temperature at radius r; °F
T_i	Temperature at inner surface; °F
T_o	Temperature at outer surface; °F
u	Displacement; ft
α	Coefficient of thermal expansion; $in/in \cdot °F$
ε	Strain; ft/ft
ε_r	Radial strain; ft/ft
ε_h	Hoop strain; ft/ft
ε_z	Axial strain; ft/ft
ν	Poisson's ratio; dimensionless
ρ	Mass density; lb_m/ft^3
σ	Stress; lb_f/ft^2

σ_h Hoop stress; lb_f/ft^2

σ_r Radial stress; lb_f/ft^2

σ_z Axial stress; lb_f/ft^2

INTRODUCTION

The importance of thermal stresses as the significant and often dominant design-feature in the stress analysis and thermo-mechanical design of pressure vessel, nuclear power reactor and more conventional heat transfer equipments, is well recognized. The determination of thermal stresses in addition to load stresses for the systems in which severe temperature gradient exists, is essential for their design for adequate strength. Nuclear structures, such as nuclear fuel elements and assembly; and bodies heated by electrical induction heaters are subjected to thermal stresses with internal heat generation. If free expansion or contraction of all the fibers of a body is permitted, no stress is caused by the change in temperature. However, when the temperature-rise in a homogeneous body is not uniform, different elements of the body tend to expand by different amounts and each element expands by an amount proportional to the local temperature-rise. Thus, the various elements exert upon each other a restraining action resulting in continuous unique displacements at every point. The system of strains produced by this restraining action cancels out all, or part of, the free thermal expansions at every point so as to ensure continuity of displacement. This system of strains must be accompanied by a corresponding system of self-equilibrating stresses. These stresses are known as thermal stresses.

Thermal stresses are induced in a structure made of dissimilar materials and subjected to thermal gradient. Also, in the case of uniform temperature-change in a homogenous body with external restrains, thermal stresses are produced.

The problem of elastic thermal stresses with internal heat generation is of great practical importance in the nuclear field and induction heating. Nuclear reactors through a chain reaction consequently produce heat, neutrons, and fission products. Plutonium and uranium are the most important nuclear fuels used in the reactors and the nuclear arms. Cladding materials, such as stainless steel and aluminum are used because of their high thermal conductivity, in order to reduce the thermal stresses among other reasons. Some alloys are also used, instead. Analytical study to determine the elastic thermal stress distributions in the aforesaid structures will yield valuable information and enable design engineers to develop criteria to avoid serious structural failure of fuel elements and assembly.

Reviews of literature on the study of thermal stresses developed in bodies subjected to temperature gradient show that considerable efforts have been directed to this area during the last two decades. Unfortunately, no reference of any published work related to this present study was available in the various reference sources. Yen and Kirmser (1) presented a solution for the determination of thermal stresses in a finite cylinder heated axisymmetrically over the curved surface.

The solution is obtained by constructing the thermoelastic displacement potential and the biharmonic Love function. Yang and Lee (2) gave a solution based on three dimensional linear theory of thermoelasticity with appropriate approximations by neglecting small terms and using St. Venant's principle. Cheung, Chen and Thirumalai (3) derived the thermal stresses in a sphere by superposing a particular displacement potential function on Boussinesq solutions. Takeuti and Noda (4), Faupel (5), Timoshenko and Woinowsky-Krieger (6), Boley and Weiner (7), and Gatewood (8) presented the solutions for the determination of thermal stresses, but internal heat generation was not taken into consideration. Bhaduri (9) developed simplified dimensionless thermal stress functions for thick, cylindrical and spherical shells subjected to temperature gradient to facilitate the design computations.

The present study is directed to the analysis and development of dimensionless elastic thermal stress functions for thick cylindrical and spherical shells with internally uniform heat generation. The analysis follows the classical approach: viz., the determination of temperature distribution in the thick shells due to internal heat generation with appropriate boundary conditions, and the consideration of the elastic-stress-strain relations for the shells, the compatibility conditions, and the equilibrium relations. The steady state temperature solutions are used to determine the radial, tangential and, when appropriate, the longitudinal thermal stresses.

THEORETICAL DEVELOPMENT AND ANALYSIS

The objective of the present study is to study the elastic thermal stresses in thick and homogenous cylindrical and spherical shells with internal heat generation. The basic governing equations for the temperature distribution in the shells are derived first and then used in the equilibrium equations to obtain the thermal stresses. The cylindrical shell is considered first.

Thick Cylindrical Shell

Cylindrical Shell (Insulated Inside)

A thick cylindrical shell, Fig. 2.1, with inner radius r_i and outer radius r_o and insulated at the inner wall and with an internal heat generation at a rate of H per unit volume is considered. The cylinder is long and its ends are considered unrestrained. The temperature T_o at the outer surface is assumed to be constant. The difference T of temperatures at any local radial location and that of the outer surface is given by

$$T = t - T_o = \frac{H}{2k}\left[\left(\frac{r_o^2 - r^2}{2}\right) - r_i^2 \ln \frac{r_o}{r}\right] \qquad (1)$$

The derivation of temperature distribution Eq. (1) is shown in the Appendix A, Eq. (5).

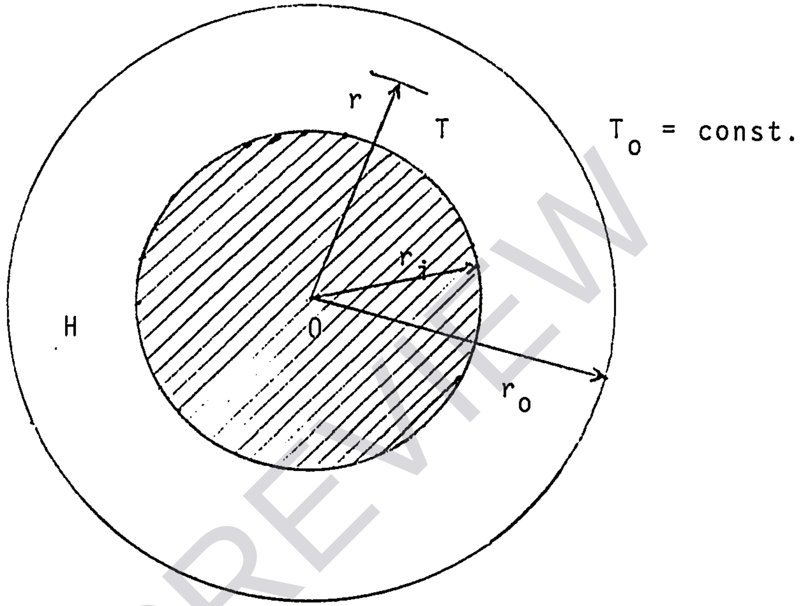

Fig. 2.1

Cylindrical Shell Section (Insulated Inside)

Figure 2.2 shows the section of a cylindrical shell. Axial strain developed as a result of the stress is uniform and constant. The radial and hoop strains (5) are given by

$$\varepsilon_r = \frac{du}{dr} \tag{2}$$

$$\varepsilon_h = \frac{u}{r} \tag{3}$$

The total strain is made up of a strain dependent on the induced stresses and a strain due to free thermal expansion; thus

$$\varepsilon_h = \frac{1}{E}\left[\sigma_h - \nu(\sigma_r + \sigma_z)\right] + \alpha(\Delta T) = \frac{u}{r} \tag{4}$$

$$\varepsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_h + \sigma_z)\right] + \alpha(\Delta T) = \frac{du}{dr} \tag{5}$$

$$\varepsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\delta_h - \sigma_r)\right] + \alpha(\Delta T) = 0 \tag{6}$$

where α = coefficient of thermal expansion, E = Young's modules, ν = Poisson's ratio.

Equilibrium equations (5) are given by

$$r\frac{d\sigma_r}{dr} + \sigma_r - \sigma_h = 0 \tag{7}$$

$$\varepsilon_r = \frac{d}{dr}(r\varepsilon_h) \tag{8}$$

$$r\frac{d}{dr}\varepsilon_r + \varepsilon_h - \varepsilon_r = 0 \tag{9}$$

Substituting Eqs. (4) and (5) into Eq. (9) and using Eq. (7) yields

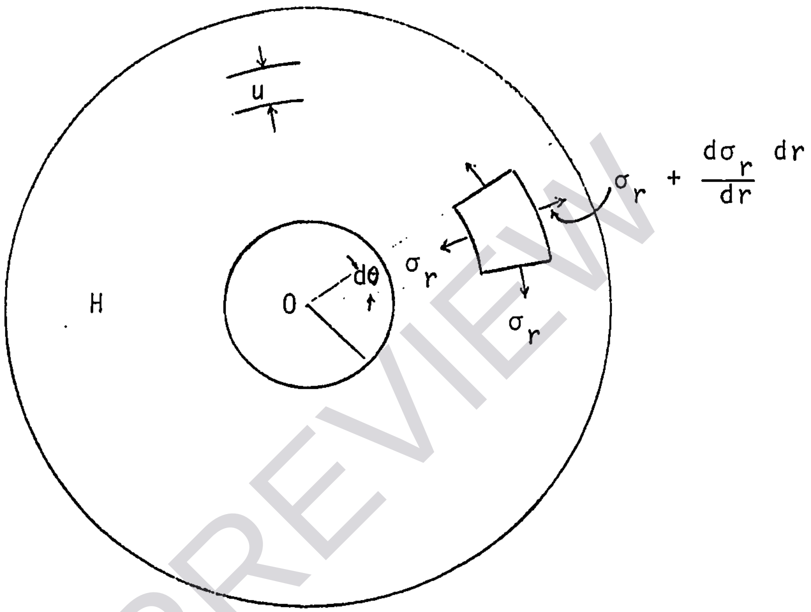

Fig. 2.2

Cylindrical Shell Section (Insulated Inside)
Showing the Stresses

$$\frac{d^2}{dr^2}\sigma_r + \frac{3}{r}\frac{d}{dr}\sigma_r = \frac{1}{r^3}\frac{d}{dr}\left(r^3 \frac{d}{dr}\sigma_r\right) =$$

$$\left(-\frac{\alpha E}{1-\nu}\right)\left(\frac{1}{r}\right)\left(\frac{dT}{dr}\right) \tag{10}$$

Solution of Eq. (10) with the boundary condition $\sigma_r = 0$ at the outside and inside surfaces, $r = r_o$ and $r = r_i$, for a hollow cylinder gives hoop stress σ_h, radial stress σ_r and the axial stress respectively by the following relations:

$$\sigma_h = \frac{\alpha E}{1-\nu}\left(\frac{1}{r^2}\right)\left(\frac{r^2 + r_i^2}{r_o^2 - r_i^2}\int_{r_i}^{r_o} Tr\,dr + \int_{r_i}^{r} Tr\,dr - Tr^2\right) \tag{11}$$

$$\sigma_r = \frac{\alpha E}{1-\nu}\left(\frac{1}{r^2}\right)\left(\frac{r^2 - r_i^2}{r_o^2 - r_i^2}\int_{r_i}^{r_o} Tr\,dr - \int_{r_i}^{r} Tr\,dr\right) \tag{12}$$

$$\sigma_z = \frac{\alpha E}{1-\nu}\left(\frac{2}{r_o^2 - r_i^2}\int_{r_i}^{r_o} Tr\,dr - T\right) \tag{13}$$

Substituting T from Eq. (1) into Eqs. (11) through (13) and integrating by parts:

$$\sigma_r = \frac{\alpha E H}{16k(1-\nu)r^2}\Bigg[\frac{r^2 - r_i^2}{r_o^2 - r_i^2}\left(r_o^4 - 4r_i^2 r_o^2 + 3r_i^4 + 4r_i^4 \ln\frac{r_o}{r_i}\right)$$

$$-\left(2r^2 r_o^2 - r^4 - 4r_i^2 r^2 \ln\frac{r_o}{r} - 2r_i^2 r^2 - 2r_i^2 r_o^2 + 3r_i^4 + 8r_i^4 \ln\frac{r_o}{r_i}\right)\Bigg] \tag{14}$$

$$\sigma_h = \frac{\alpha E H}{16k(1-\nu)r^2}\left[\frac{r^2 + r_i^2}{r_o^2 - r_i^2}(r_o^4 - 4r_i^2 r_o^2 + 3r_i^4 + 4r_i^4 \ln\frac{r_o}{r_i}\right.$$

$$-2(2r^2 r_o^2 - r^4 - 4r_i^2 r^2 \ln\frac{r_o}{r} - 2r_i^2 r^2 - 2r_i^2 r_o^2 +$$

$$\left.3r_i^4 + 4r_i^4 \ln\frac{r_o}{r_i} - r^2(4r_o^2 - 4r^2 - 8r_i^2 \ln\frac{r_o}{r})\right] \qquad (15)$$

$$\sigma_z = -\frac{\alpha E H}{16k(1-\nu)}\left[4r_o^2 - 4r^2 - 8r_i^2 \ln\frac{r_o}{r} - \frac{2}{r_o^2 - r_i^2}\cdot\right.$$

$$\left.(r_o^4 - 4r_i^2 r_o^2 + 3r_i^4 + 4r_i^4 \ln\frac{r_o}{r})\right] \qquad (16)$$

Substituting $R = \frac{r}{r_i}$ and $R_o = \frac{r_o}{r_i}$ in the above equations, normalized stress functions F_r, F_h, and F_z are obtained:

$$F_r = \frac{\sigma_r}{\frac{\alpha E H r_i^2}{16k(1-\nu)}} = \left[\frac{R^2 - 1}{R^2(R_o^2 - 1)}(4 R_o^4 \ln R_o) - \frac{1}{R^2}(2 R^2 R_o^2 -\right.$$

$$R^4 - 4 R^2 \ln\frac{R_o}{R} - 2 R^2 - 2 R_o^2 + 3 + 8 \ln R_o)$$

(17)

$$F_h = \frac{\sigma_H}{\frac{\alpha E H r_i^2}{16k(1-\nu)}} = \Bigg[\frac{R^2 + 1}{R^2(R_o^2 - 1)} (R_o^4 - r\ R_o^2 + 3 +$$

$$4 \ln R_o) - \frac{2}{R^2} (2\ R^2 R_o^2 - R^4 - 4\ R^2 \ln \frac{R_o}{R} -$$

$$- 2\ R^2 - 2R_o^2 + 3 + 4 \ln R_o)$$

$$- (4\ R_o^2 - 4\ R^2 - 8 \ln \frac{R_o}{R}) \Bigg] \tag{18}$$

$$F_z = \frac{\sigma_Z}{\frac{\alpha E H r_i^2}{16k(1-\nu)}} = - \Bigg[r\ R_o^2 - r\ R^2 - 8 \ln \frac{R_o}{R} -$$

$$\frac{2}{(R_o^2 - 1)} (R_o^4 - 4\ R_o^2 + 3 + 4 \ln \frac{R_o}{R}) \Bigg] \tag{19}$$

<u>Cylindrical Shell (Insulated Outside</u>

In this case, Fig. 2.3, with the outside wall insulated, the temperature difference T between the temperature at any local radial point and the temperature at the inner surface is given by

$$T = t - T_i = \frac{H}{2k} \left[r_o^2 \ln \frac{r}{r_i} - \frac{r^2 - r_i^2}{2} \right] \tag{20}$$

The derivation of Eq. (20) is given in Appendix A, Eq. (9).

Substituting T from Eq. (20) into Eqs. (11) through (13) and integrating by parts, one gets

$$r = \frac{\alpha E H}{16k(1-\nu) r^2} \Bigg[\frac{r^2 - r_i^2}{r_o^2 - r_i^2} (4r_o^4 \ln \frac{r_o}{r_i} - 3r_o^4 +$$

$$4r_i^2 r_o^2 - r_i^4) + 2r_o^2 r^2 (1 - 2 \ln \frac{r}{r_i}) -$$

$$2\ r_i^2 (r_o^2 + r^2) + r^4 + r_i^4 \Bigg] \tag{20}$$

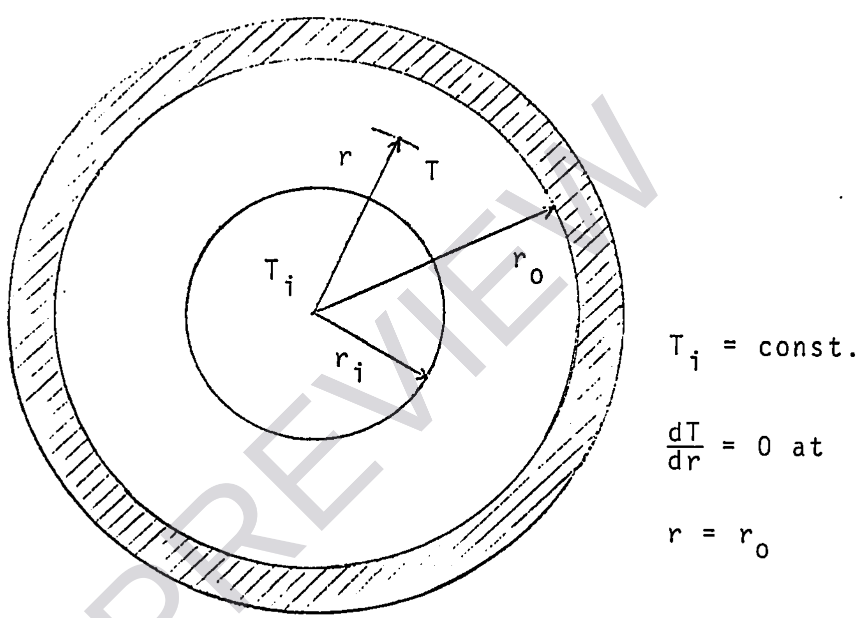

Fig. 2.3

Cylindrical Shell Section (Insulated Outside)

$$\sigma_h = \frac{\alpha EH}{16k(1-\nu)r^2}\left[\frac{r^2 + r_i^2}{r_o^2 - r_i^2}\left(4r_o^4 \ln\frac{r_o}{r_i} - 3r_o^4 +\right.\right.$$

$$\left.4r_i^2 r_o^2 - r_i^4\right)$$

$$-2\, r_o^2 r^2 \left(1 + 2 \ln\frac{r}{r_o}\right) + 2\, r_i^2 \left(r_o^2 - r^2\right) +$$

$$\left.3r^4 - r_i^4\right] \qquad (21)$$

and

$$\sigma_z = \frac{\alpha EH}{16k(1-\nu)r^2}\left[8r_o^2 \ln\frac{r}{r_o} - 4r^2 + 4r_i^2 - \frac{2}{r_o^2 - r_i^2}\right.$$

$$\left.\left(4r_o^4 \ln\frac{r_o}{r_i} - 3r_o^4 + 4r_i^2 r_o^2 - r_i^4\right)\right] \qquad (22)$$

Substituting $R = \frac{r}{r_i}$ and $R_o = \frac{r_o}{r_i}$ in the above equations, normalized stress functions F_r, F_h, and F_z are obtained as follows:

$$F_r = \frac{\sigma_r}{\frac{\alpha EHr_i^2}{16k(1-\nu)}} = \left[\frac{R^2 - 1}{R^2(R_o^2 - 1)}\left(4\, R_o^4 \ln R_o - 3R_o^4 +\right.\right.$$

$$\left.4R_o^2 - 1\right) + 2\, R_o^2(1 - 2 \ln R)$$

$$\left.- \frac{2}{R^2}\left(R_o^2 + R^2\right) + R^2 + \frac{1}{R^2}\right] \qquad (23)$$

$$F_h = \frac{h}{\frac{EHr_i^2}{16k(1-\ )}} = \left[\frac{R^2 + 1}{R^2(R_o^2 - 1)}\left(4\, R_o^4 \ln R_o - 3\, R_o^4 +\right.\right.$$

$$\left.4R_o^2 - 1\right) - 2\, R_o^2 \left(1 + 2 \ln\frac{R}{R_o}\right)$$

$$\left.+ \frac{2}{R^2}\left(R_o^2 - R^2\right) + 3R^2 - \frac{1}{R^2}\right] \qquad (24)$$