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SOLUTIONS TO BOUNDARY VALUE PROBLEMS AND PERIODIC SOLUTIONS OF SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS

by

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SOLUTIONS TO BOUNDARY VALUE PROBLEMS AND PERIODIC SOLUTIONS OF

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Klaus Schmitt

Contents

1. Introduction

Consider the second-order differential equation

$$x'' = f(x,x',t) \tag{1.1}$$

where $f(x,x',t)$ is a real valued function defined on one of the following regions

$A = \{(x,x',t) : |x| + |x'| < +\infty,\ a \leq t \leq b,$ where a and b are finite$\}$

$B = \{(x,x',t) : |x| + |x'| < +\infty,\ a \leq t < +\infty,$ where a is finite$\}$

$C = \{(x,x',t) : |x| + |x'| + |t| < +\infty\}.$

If $f(x,x',t)$ is defined on A we shall be interested in finding solutions of (1.1) which are periodic on the compact interval $[a,b]$, and also in solving the boundary value problems (BVP's)

$$x'' = f(x,x',t), \quad x(a) = c, \quad x(b) = d \tag{1.2}$$

and

$$\begin{aligned} &x'' = f(x,x',t) \\ &a_o x(a) + a_1 x'(a) = c, \quad |a_o| + |a_1| > 0 \\ &b_o x(b) + b_1 x'(b) = d, \quad |b_o| + |b_1| > 0. \end{aligned} \tag{1.3}$$

For functions $f(x,x',t)$ defined on B we shall seek solutions of the BVP

$$x'' = f(x,x',t), \quad x(a) = c. \tag{1.4}$$

In case $f(x,x',t)$ is defined on C we shall investigate the existence of solutions of (1.1) which are defined for all t with $|t| < +\infty$.

Cesari in [5] has studied the problem of the existence of periodic solutions for the first-order system $z' = F(z,t)$, where $z \in R^n$, and $F(z,t)$ is a function from $R^n \times [a,b]$ to R^n which is required to satisfy a Lipschitz condition (L-condition) with respect to z. Motivated by the work of Cesari in [5], Barbălat in [2], and Massera in [16], Knobloch in [12], [13], and [14] establishes existence theorems for periodic solutions of first-order systems and more specifically of second order ordinary differential equations, his method of proof being primarily based on the work of Cesari in [5]. A L-condition therefore is an essential hypothesis in his existence theorems.

Using a theorem of Jackson and Schrader in [10], we are able to obtain results similar to those of Knobloch in [14]. Our results, however, improve the results of Knobloch in [14] considerably in the sense that we no longer impose a L-condition on $f(x,x',t)$.

Our most important results about periodic solutions of (1.1) are contained in Theorems 3.2 and 3.4 and their corollaries. We state Theorem 3.4 in a somewhat less general form than given in the work to follow.

<u>Theorem</u>: Let $f(x,x',t) \in C(R^2 \times [a,b])$. Assume that there exist functions $\alpha(t), \beta(t) \in C^2[a,b]$ satisfying the following properties

i) $\alpha''(t) \geq f(\alpha(t),\alpha'(t),t)$, $\beta''(t) \leq f(\beta(t),\beta'(t),t)$

ii) $\alpha(a) = \alpha(b)$, $\alpha'(a) \geq \alpha'(b)$

$\beta(a) = \beta(b)$, $\beta'(a) \leq \beta'(b)$

iii) $\alpha(t) \leq \beta(t)$

for all $a \leq t \leq b$. Further assume that there exist functions $\phi(x,t)$, $\psi(x,t)$ satisfying the properties

iv) $\phi(x,t)$, $\psi(x,t)$ are defined and continuously differentiable on the set

$\omega = \{(x,t) : \alpha(t) \leq x \leq \beta(t),\ a \leq t \leq b\}$

v) $\phi(x,a) = \phi(x,b)$, $\psi(x,a) = \psi(x,b)$

vi) $\phi(x,t) \leq \psi(x,t)$, for all $(x,t) \in \omega$

vii) the functions $f(x,\psi,t) - \psi_x\psi - \psi_t$ and

$f(x,\phi,t) - \phi_x\phi - \phi_t$ do not change sign on ω

viii) $\phi(\alpha,t) \leq \alpha' \leq \psi(\alpha,t)$, $\phi(\beta,t) \leq \beta' \leq \psi(\beta,t)$,

for all $a \leq t \leq b$.

Then the differential equation (1.1) has a periodic solution $x(t)$ such that $(x(t),t) \in \omega$ and $\phi(x(t),t) \leq x'(t) \leq \psi(x(t),t)$ for all $a \leq t \leq b$.

The requirement of the existence of such functions $\phi(x,t)$ and $\psi(x,t)$ has the following geometric interpreta-

tion. If $x(t)$ is a periodic solution of (1.1), then the trajectory $(x(t),x'(t),t)$ can cross the surfaces $(x,\phi(x,t),t)$ and $(x,\psi(x,t),t)$ in one direction only.

Theorem 4.2 is our major result concerning the BVP (1.2). Again in a somewhat less general form than to follow, it can be stated as follows.

<u>Theorem</u>: Let $f(x,x',t) \in C(R^2 \times [a,b])$ be such that the initial value problem (IVP)

(1.5) $\quad x'' = f(x,x',t), \quad x(t_0) = x_0, \quad x'(t_0) = x_0'$

is locally uniquely solvable for every $t_0 \in [a,b]$. Let there exist functions $\alpha(t),\beta(t) \in C^2[a,b]$ satisfying properties i) and iii) of the theorem above such that either $\alpha(a) = c$, $\alpha(b) = d$ or $\beta(a) = c$, $\beta(b) = d$. Assume there exist functions $\phi(x,t),\psi(x,t)$ satisfying properties iv), vi), and viii) of the above theorem and

vii)' the functions $f(x,\phi,t) - \phi_x\phi - \phi_t$ and $f(x,\psi,t) - \psi_x\psi - \psi_t$ do not assume the value zero for any $(x,t) \in \omega$.

Then the BVP (1.2) has a solution $x(t)$ such that $(x(t),t) \in \omega$ and $\phi(x(t),t) \leq x'(t) \leq \psi(x(t),t)$.

The BVP (1.3) has been studied by Ehrmann [7] and Keller [11]. Each of these authors gives sufficient conditions that the BVP (1.3) have a solution. Our result

concerning this BVP is essentially due to Keller, who however supplies an incorrect proof. Using results concerning the BVP (1.2) due to Fountain and Jackson [8] and Bebernes [3], we nevertheless succeed in supplying a proof. The result can be stated as follows.

<u>Theorem</u>: Let $f(x,x',t)$ be continuous on the set A and assume that $f(x,x',t)$ has continuous derivatives with respect to x and x' which satisfy $\frac{\partial}{\partial x} f(x,x',t) \geq 0$, $\left|\frac{\partial}{\partial x'} f(x,x',t)\right| \leq M$, for some $M \geq 0$ and all $a \leq t \leq b$. Then the BVP (1.3) has a unique solution in case a_o and a_1, b_o and b_1 have the same sign and $|a_o| + |b_o| > 0$.

2. Preliminary Results

In the work to follow we shall always denote the compact interval $[a,b]$ by I and its interior (a,b) by I^o. A partition $a = t_o < t_1 < t_2 < \ldots < t_n = b$ of I will be denoted by P_n.

Definition 2.1: A real-valued function $F(z,t)$ is said to belong to class $C_p(R^m \times I)$ in case there exists a partition P_n of I such that either $F(z,t) \in C(R^m \times [t_{i-1},t_i))$ or $F(z,t) \in C(R^m \times (t_{i-1},t_i])$ and $F(z,t)$ can be continuously extended to $R^m \times [t_{i-1},t_i]$ for $i = 1,2,\ldots,n$. $F(z,t)$ is said to belong to class $C_p^k(R^m \times I)$, $k \geq 1$, in case $F(z,t) \in C^{k-1}(R^m \times I)$ and the k^{th} partial derivatives of $F(z,t)$ exist and belong to class $C_p(R^m \times I)$.

Definition 2.2: A real-valued function $g(t)$ is said to belong to class $C_p(I)$ in case there exists a partition P_n of I such that $g(t) \in C[t_{i-1},t_i)$ or $g(t) \in C(t_{i-1},t_i]$ and $g(t)$ can be continuously extended to $[t_{i-1},t_i]$ for $i = 1,2,\ldots,n$. $g(t)$ is said to belong to class $C_p^k(I)$, $k \geq 1$, in case $g(t) \in C^{k-1}(I)$ and $g^{(k)}(t)$ belongs to class $C_p(I)$.

The following well-known result is usually stated for $f(x,x',t) \in C(R^2 \times I)$. In our work, however, we shall need this theorem for $f(x,x',t) \in C_p(R^2 \times I)$. We therefore include a proof.

Theorem 2.1: Let $f(x,x',t) \in C_p(R^2 \times I)$. Let $M > 0$, $N > 0$ be given and let $Q = \sup_{\substack{a \leq t \leq b \\ |x| \leq 2M \\ |x'| \leq 2N}} |f(x,x',t)|$ and $\delta = \min\{\sqrt{\frac{8M}{Q}}, \frac{2N}{Q}\}$. Then for any interval $[t^1,t^2] \subseteq I$, with $|t^1-t^2| \leq \delta$, and any x_1, x_2 such that $|x_2-x_1/t^1-t^2| \leq N$, $|x_1| \leq M$, $|x_2| \leq M$, there exists a solution $x(t)$ of (1.1) on $[t^1,t^2]$ with $x(t) \in C_p^2[t^1,t^2]$ and $x(t^1) = x_1$, $x(t^2) = x_2$.

Proof: Let $X = C'([t^1,t^2])$ and for $h \in X$ define $||h|| = \sup_{t^1 \leq t \leq t^2} |h(t)| + \sup_{t^1 \leq t \leq t^2} |h'(t)|$. Let $H = \{h \in X :$ $|h(t)| \leq 2M$, $|h'(t)| \leq 2N$, for all $t^1 \leq t \leq t^2\}$. Then H is a closed and bounded convex subset of the Banach space $(X, || \cdot ||)$.

Define the mapping T on H in the following way. If $h \in H$ let Th be the function which at t has the value

$$Th(t) = \int_{t^1}^{t^2} G(t;s)f(h(s),h'(s),s)ds + \ell(t),$$ where $G(t;s)$

is the Green's function associated with the BVP $x'' = 0$, $x(t^1) = 0 = x(t^2)$, and $\ell(t)$ is the linear function through the points (x_1,t^1) and (x_2,t^2). If now $|t^2-t^1| \leq \delta$, $|x_1-x_2/t^1-t^2| \leq N$, $|x_1| \leq M$, $|x_2| \leq M$, then T maps H back into itself. For

$$|Th(t)| \leq Q \int_{t^1}^{t^2} |G(t;s)|ds + M \leq \frac{(t^1-t^2)^2}{8} Q + M$$

and

$$|(Th)'(t)| \leq Q \int_{t^1}^{t^2} |G_t(t;s)|ds + N \leq \frac{t^2-t^1}{2} Q + N$$

which implies that $|Th(t)| \leq 2M$ and $|(Th)'(t)| \leq 2N$. For continuity of T on H it suffices to show that if $\{h_n\}_{n=1}^{\infty} \subseteq H$ and $\{h_n\}_{n=1}^{\infty}$ converges in norm to h, then $\{Th_n\}_{n=1}^{\infty}$ is Cauchy in norm. Since $f(x,x',t) \in C_p(R^2 \times I)$ there exists a partition P_n of I such that f can be continuously extended to $R^2 \times [t_{i-1},t_i]$, and thus f has a uniformly continuous extension to the compact set $[-2M,2N] \times [-2N,2N] \times [t_{i-1},t_i]$, for $i = 1,2,\ldots,n$. Hence for $\varepsilon > 0$ there exists $\rho > 0$ such that $|f(x_1,x_1',t) - f(x_2,x_2',t)| < \varepsilon$ for

all (x_i, x_i', t) with $|x_1 - x_2| < \rho$, $|x_1' - x_2'| < \rho$, $|x_i| \leq 2M$, $|x_i'| \leq 2N$, $i = 1,2$, and $a \leq t \leq b$. Since $\{h_n\}_{n=1}^{\infty}$ is Cauchy in norm, we have that for given $\rho > 0$ there exists $N_\rho > 0$ such that $n,m \geq N_\rho$ implies $|h_n(t) - h_m(t)| < \rho$ and $|h_n'(t) - h_m'(t)| < \rho$ for all $t^1 \leq t \leq t^2$. Therefore if $n,m \geq N_\rho$, $|Th_n(t) - Th_m(t)| \leq \varepsilon \frac{(t^1-t^2)^2}{8}$ and $|(Th_n)'(t) - (Th_m)'(t)| \leq \varepsilon \frac{|t^1-t^2|}{2}$. Thus $\{Th_n\}_{n=1}^{\infty}$ is Cauchy in norm.

We next show that $T(H)$ is sequentially compact. Let $\{g_n\}_{n=1}^{\infty} \subseteq T(H)$, then there exists $\{h_n\}_{n=1}^{\infty} \subseteq H$ such that $g_n = Th_n$, for all $n \geq 1$. Both $\{g_n\}_{n=1}^{\infty}$ and $\{g_n'\}_{n=1}^{\infty}$ are uniformly bounded families, and since $g_n(t) \varepsilon C_p^2[t^1, t^2]$ both families are also equicontinuous. Therefore by the Ascoli-Arzela Theorem there exist subsequences $\{g_{n_k}\}_{k=1}^{\infty}$ and $\{g_{n_k}'\}_{k=1}^{\infty}$ which converge uniformly on $[t^1, t^2]$. Hence there exists a subsequence of $\{g_n\}_{n=1}^{\infty}$ which converges uniformly in norm. Hence $T(H)$ is sequentially compact and thus $\overline{T(H)}$ is compact. By a corollary to the Schauder-Tychonoff Fixed Point Theorem, see [6; p. 456], there exists $x \varepsilon H$ such that $Tx = x$. We further have that

$x''(t) = f(x(t), x'(t), t)$ at all points of continuity of $f(x, x', t)$, and $x(t^1) = x_1$, $x(t^2) = x_2$. Thus $x(t)$ is the the desired solution of (1.1).

Corollary 2.1: Let $f(x, x', t) \varepsilon C_p(R^2 \times I)$. Let there exist constants C_1 and C_2 such that $|f(x, x', t)| \leq C_1 + C_2|x'|^{1/2}$, then the BVP (1.2) always has a solution.

Proof: If $C_1 = 0 = C_2$ the corollary is trivially true. Thus assume that C_1 and C_2 are not both zero. Choose $M = N$ large enough so that $|c| \leq M$, $|d| \leq M$,

$|c\text{-}d/b\text{-}a| \leq M$, $b-a \leq \sqrt{\dfrac{8M}{C_1 + C_2(2M)^{1/2}}}$ and

$b-a \leq \dfrac{2M}{C_1 + C_2(2M)^{1/2}}$. It follows that $Q \leq C_1 + C_2(2M)^{1/2}$

and hence $\sqrt{\dfrac{8M}{Q}} \geq b-a$ and $2M/Q \geq b-a$. Now apply Theorem 2.1.

The following result in a somewhat less general form is due to Jackson and Schrader. The proof of the result in the form given below requires only minor modifications of the proof given by Jackson and Schrader [10; Theorem 3.1]. We nevertheless shall present a proof which in turn will give us the result in the more general setting needed in this work.

However, we first shall need the following definitions.

Definition 2.3: A function $\alpha(t)$ is said to be a lower solution of (1.1) in case $\alpha(t) \varepsilon C_p^2(I)$ and $\alpha''(t) \geq f(\alpha(t),\alpha'(t),t)$, for all $a \leq t \leq b$.

Definition 2.4: A function $\beta(t)$ is said to be an upper solution of (1.1) in case $\beta(t) \varepsilon C_p^2(I)$ and $\beta''(t) \leq f(\beta(t),\beta'(t),t)$, for all $a \leq t \leq b$.

In the work to follow, the symbols α and β shall always denote lower and upper solutions respectively. The symbols ω and Ω will be used exclusively for the following sets

$$\omega = \{(x,t) : \alpha(t) \leq x \leq \beta(t),\ t \varepsilon I\}$$

$$\Omega = \{(x,x',t) : (x,t) \varepsilon \omega,\ |x'| < +\infty\}.$$

Definition 2.5: Let $\alpha(t) \leq \beta(t)$ for all $t \varepsilon I$. The function $f(x,x',t) \varepsilon C_p(R^2 \times I)$ is said to satisfy a Nagumo condition with respect to the pair $\alpha(t),\beta(t)$ in case there exists a positive continuous function $h(s)$, $0 \leq s < +\infty$, such that $|f(x,x',t)| \leq h(|x'|)$ for all $(x,x',t) \varepsilon \Omega$ and

(2.1) $$\int_\lambda^\infty \frac{sds}{h(s)} > \sup_{t\varepsilon I} \beta(t) - \inf_{t\varepsilon I} \alpha(t), \text{ where}$$

$$\lambda = \max\{|\alpha(a)-\beta(b)/b-a|, |\alpha(b)-\beta(a)/b-a|\}.$$

Lemma 2.1: Let $\alpha(t) \leq \beta(t)$, for all $t \varepsilon I$ and $f(x,x',t) \varepsilon C_p(R^2 \times I)$. Let $f(x,x',t)$ satisfy a Nagumo condition with respect to the pair $\alpha(t),\beta(t)$. Then there exists a positive constant M, such that, whenever

$x(t) \varepsilon C_p^2(I)$ is a solution of (1.1) with $(x(t),t) \varepsilon \omega$ for all $t \varepsilon I$, $|x'(t)| \leq M$ for all $t \varepsilon I$.

<u>Proof</u>: Let $x(t)$ be a solution of (1.1) with $(x(t),t) \varepsilon \omega$ for all $t \varepsilon I$. Let $\lambda_o = |x(a)-x(b)/b-a|$. In view of (2.1) there exists a positive constant M such that

$$\int_{\lambda_o}^{M} \frac{sds}{h(s)} > \sup_{t\varepsilon I} \beta(t) - \inf_{t\varepsilon I} \alpha(t).$$

There always exists $t_o \varepsilon I$ such that $|x'(t_o)| = \lambda_o$. Let $|x'(t)|$ assume its maximum at $t = t_1$. We may assume that $|x'(t_1)| > \lambda_o$. Several cases must be considered, depending on whether $t_1 < t_o$ or $t_1 > t_o$ and whether $x'(t_1) < 0$ or $x'(t_1) > 0$. To treat a specific case, assume that $t_1 > t_o$, $x'(t_1) > 0$ and $x'(t_o) = \lambda_o$ and $\lambda_o < x'(t)$ for $t \varepsilon (t_o,t_1]$. Now $x''(t) \leq |f(x,x',t)| \leq h(|x'|)$ on $[t_o,t_1]$. Hence

$$\int_{t_o}^{t_1} \frac{x''(t)x'(t)}{h(|x'|)} dt \leq \int_{t_o}^{t_1} x'(t)dt = x(t_1) - x(t_o)$$

$\leq \sup_{t\varepsilon I} \beta(t) - \inf_{t\varepsilon I} \alpha(t)$. Making a change of variables, we get

$$\int_{\lambda_o}^{x'(t_1)} \frac{sds}{h(s)} \leq \sup_{t\varepsilon I} \beta(t) - \inf_{t\varepsilon I} \alpha(t).$$

This implies that $x'(t_1) < M$. The other cases can be

treated similarly.

<u>Theorem 2.2</u>: Let $f(x,x',t) \varepsilon C_p(R^2 \times I)$, and assume there exists a lower solution $\alpha(t)$ and an upper solution $\beta(t)$ of (1.1) with $\alpha(t) \leq \beta(t)$ for all $t \varepsilon I$. Let $f(x,x',t)$ satisfy a Nagumo condition with respect to the pair $\alpha(t),\beta(t)$. Then the BVP (1.2) has a solution $x(t)$ with $(x(t),t) \varepsilon \omega$, for all $t \varepsilon I$, for any c,d with $\alpha(a) \leq c \leq \beta(a)$, $\alpha(b) \leq d \leq \beta(b)$.

<u>Proof</u>: Choose $N > 0$ large enough so that $|\alpha'(t)| \leq N$, $|\beta'(t)| \leq N$, for all $t \varepsilon I$ and $\int_\lambda^N \frac{sds}{h(s)} > \sup_{t\varepsilon I} \beta(t) - \inf_{t\varepsilon I} \alpha(t)$. Define $F(x,x',t)$ by

$$F(x,x',t) = \begin{cases} f(x,x',t), & \text{if } (x(t),t)\,\varepsilon\,\omega,\ |x'| \leq N \\ f(x,N,t), & \text{if } (x(t),t)\,\varepsilon\,\omega,\ x' > N \\ f(x,-N,t), & \text{if } (x(t),t)\,\varepsilon\,\omega,\ x' < -N \end{cases}$$

and extend $F(x,x',t)$ to all of $R^2 \times I$ by setting

$$F(x,x',t) = \begin{cases} F(\beta(t),x',t) + (x-\beta(t))^{1/2}, & \text{if } x > \beta(t) \\ F(\alpha(t),x',t) - (\alpha(t)-x)^{1/2}, & \text{if } x < \alpha(t). \end{cases}$$

$F(x,x',t)$ so defined belongs to class $C_p(R^2 \times I)$, and $|F(x,x',t)| \leq C_1+C_2|x|^{1/2}$, where

$C_1 = \sup\{|f(x,x',t)| : (x,t) \varepsilon\, \omega,\ |x'| \leq N\} + \sup_{t\varepsilon I} |\alpha(t)|^{1/2}$

$+ \sup_{t\varepsilon I} |\beta(t)|^{1/2}$ and $C_2 = 1$.

Using Corollary 2.1 we obtain that the BVP

(2.2) $$x'' = F(x,x',t)$$
$$x(a) = c,\ x(b) = d$$

has a solution $x_o(t)\ \varepsilon\ C_p^2(I)$.

We claim that $(x_o(t),t)\ \varepsilon\ \omega$ for all $t\ \varepsilon\ I$. Assume there exists an interval $[t_1,t_2] \subseteq [a,b]$ such that $x_o(t_i) = \beta(t_i)$, $i = 1,2$, and $x_o(t) > \beta(t)$ on (t_1,t_2). Let $y(t) = x_o(t) - \beta(t)$ on $[t_1,t_2]$. Let $t_3\ \varepsilon\ (t_1,t_2)$ be a point where $y(t)$ assumes its positive maximum. We have that $y'(t_3) = x_o'(t_3) - \beta'(t_3) = 0$ and since $y(t)\ \varepsilon\ C_p^2(I)$, $y''(t_3+0) = \lim_{h\to 0^+} y''(t_3+h) \leq 0$ and $y''(t_3-0) = \lim_{h\to 0^-} y''(t_3+h) \leq 0$. But

$$\begin{aligned} y''(t_3\pm 0) &= x_o''(t_3\pm 0) - \beta''(t_3\pm 0) \\ &\geq x_o''(t_3\pm 0) - f(\beta(t_3),\beta'(t_3),t_3\pm 0) \\ &\geq f(\beta,\beta',t_3\pm 0) + (x_o(t_3)-\beta(t_3))^{1/2} - f(\beta,\beta',t_3\pm 0) > 0, \end{aligned}$$

This however contradicts the fact that $y(t)$ has a maximum at t_3. Hence $x_o(t) \leq \beta(t)$, for all $t \in I$. Similarly $\alpha(t) \leq x_o(t)$.

By Lemma 2.1 we further have that $|x_o'(t)| \leq N$ for all $t \in I$. Since however $F(x,x',t) = f(x,x',t)$, whenever $(x,t) \in \omega$ and $|x'| \leq N$, this implies that the solution $x_o(t)$ of (2.2) is actually a solution of (1.2).