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1. INTRODUCTION AND SUMMARY

In the "classical" reduced Hausdorff moment problem, a sequence of constants $(\mu_1, \mu_2, \ldots, \mu_k)$ and an interval $[a, b]$ $(-\infty < a < b < \infty)$ are given: a cumulative distribution function $F(x)$ is said to be a solution if

$$\int_{-\infty}^{\infty} x^i dF(x) = \mu_i, \quad i = 1, 2, \ldots, k,$$

and $F(a-0) = 0$ and $F(b) = 1$. The geometric structure of the solution set for this problem has been studied by Karlin and Shapley [4].

In the "classical" reduced Stieltjes moment problem, the interval $[a, b]$ is replaced by $[0, \infty)$.

Mulholland and Rogers [7] have studied a generalized moment problem, where the x^i are replaced by Borel-measurable functions $f_i(x)$, $i = 1, 2, \ldots, k$, and have shown that any solution $F(x)$ has a representation

$$F(x) = \int_0^1 H_t(x)dt;$$

where each $H_t(x)$ $(0 \leq t \leq 1)$, is an extreme point of the solution set, and has spectrum containing at most $k + 1$ points.

In this paper, we restrict attention to continuous $f_i(x)$, $i = 1, 2, \ldots, k$, and give constructive proofs to show that the solution sets for the generalized Hausdorff and Stieltjes moment problems have a representation as the closure of the convex hull of cumulative distribution functions with spectra containing no more than $k + 1$ points.

The principal results of Sections 4 and 7 in this paper are special cases of Theorem 2 of Mulholland and Rogers [7]. However, the methods

used in obtaining these results apply the properties of the moment space, which is of finite dimension, and therefore the processes employed are essentially those of the geometry of finite dimensional Euclidean spaces and consequently substantially simpler than the techniques used in the Mulholland and Rogers [7].

The constructive techniques of this paper may be applied to a number of problems arising in the evaluation and approximation of integrals, since the cumulative distribution functions involved are shown to be limits of sequences of cumulative distribution functions each of which have a finite number of jumps, and a process for constructing the sequence is given. On the other hand, the methods of Mulholland and Rogers, in not assuming the continuity of the functions $f_i(x)$, $i = 1, 2, \ldots, k$, require a substantially more complicated construction: (see Theorem 1, Mulholland and Rogers), since they are then unable to use the Helly-Bray Theorem in obtaining properties of the moment space. Instead, their methods are those of integration theory rather than those of the geometry of convex bodies in Euclidean k-space.

The application of the above techniques to minimizing (maximizing) integrals, and to quadrature problems is discussed.

As a statistical application, we obtain an approximate lower bound for the entropy of a multinomial population with unknown parameters.

2. THE MOMENT SPACE

FOR THE GENERALIZED REDUCED HAUSDORFF MOMENT PROBLEM

Let $\mathcal{F}$ be the space of cumulative distribution functions $F(x)$ and $\mathcal{F}_{[a,b]}$ the subset of $\mathcal{F}$ such that $F(x) \in \mathcal{F}_{[a,b]}$ if and only if $F(a-0) = 0$ and $F(b) = 1$ $(-\infty < a < b < \infty)$.

A sequence of cumulative distribution functions $\{F_n(x)\}_{n=1}^{\infty}$ is said to <u>converge in distribution</u> to a cumulative distribution function $F(x)$ if $\lim_{n \to \infty} F_n(x) = F(x)$ at every continuity point of $F(x)$; we will denote this by $F_n(x) \xrightarrow{d} F(x)$.

The space $\mathcal{F}$ is topologized with the "Helly-Bray" topology, that is, the topology induced by convergence in distribution. It is well known that $\mathcal{F}$ is a metric space under this topology. The appropriate metric is the Lévy metric described in Gnedenko and Kolmogorov [1].

For any $F(x) \in \mathcal{F}$, we say that x is in the <u>spectrum of $F(x)$</u>, that is, $x \in \mathcal{S}(F)$, if for every $\varepsilon > 0$,

$$F(x+\varepsilon) - F(x-\varepsilon) > 0.$$

Let $f_1(x), f_2(x), \ldots, f_k(x)$ be k given functions continuous on $[a, b]$ such that $1, f_1(x), f_2(x), \ldots, f_k(x)$ are linearly independent on $[a, b]$. Then for any cumulative distribution function $F(x) \in \mathcal{F}_{[a,b]}$, let

$$\int_{-\infty}^{\infty} f_i(x)dF(x) = C_i(F), \qquad i = 1, 2, \ldots, k,$$

and let

$$\tilde{C}(F) = (C_1(F), C_2(F), \ldots, C_k(F)).$$

Then $\tilde{C}(F)$ is a linear operator on $\mathcal{F}_{[a,b]}$ with $\tilde{C} : \mathcal{F}_{[a,b]} \to E^k$ (Euclidean k-space).

Let

$$Q_{[a,b]} = \{ \tilde{c} = (c_1, c_2, \ldots, c_k) : \tilde{C}(F) = (c_1, c_2, \ldots, c_k) \text{ for some } F(x) \in \mathcal{F}[a,b] \}$$

$Q_{[a,b]}$ is called the moment space. Since $\mathcal{F}_{[a,b]}$ is a closed, convex set, it follows readily that $Q_{[a,b]}$ is a closed convex set. (The proof given by Harris [3] for the case $f_i(x) = x^i$ $(i = 1, 2, \ldots, k)$ can easily be modified and applied here, since only the continuity of $f_i(x)$ $(i = 1, 2, \ldots, k)$ is involved). Since $1, f_1(x), f_2(x), \ldots, f_k(x)$ are linearly independent on $[a, b]$, it is clear that $Q_{[a,b]}$ is a k-dimensional convex set.

A cumulative distribution function $F(x)$ is said to be degenerate if there is a real number t such that $\mathcal{S}(F) = \{t\}$. Let $\mathcal{J}_{[a,b]}$ be the set of degenerate cumulative distribution functions in $\mathcal{F}_{[a,b]}$. Let $I_t(x) \in \mathcal{J}_{[a,b]}$ denote the cumulative distribution function

$$I_t(x) = \begin{cases} 0, & x < t \\ 1, & x \geq t \end{cases} \qquad (a \leq t \leq b).$$

Then

$$\tilde{C}(I_t) = (f_1(t), f_2(t), \ldots, f_k(t)).$$

Let $E(Q_{[a,b]})$ be the set of extreme points of $Q_{[a,b]}$. Then we have the following theorem.

Theorem 1. $E(Q_{[a,b]}) \subset \tilde{C}(\mathcal{J}_{[a,b]})$.

Proof. Assume that for some $F(x) \in \mathcal{F}_{[a,b]}$, $\tilde{C}(F) \in E(Q_{[a,b]})$ and $\tilde{C}(F) \notin \tilde{C}(\mathcal{J}_{[a,b]})$. Then $\mathcal{S}(F)$ contains at least two points in $[a, b]$. Let P_F be the probability measure induced by F.

If $f_i(x)$ $(i = 1, 2, \ldots, k)$ are constant a.e. (P_F), then we can choose a degenerate distribution $I(x)$ with $\mathscr{S}(I) \subset \mathscr{S}(F)$, and $\tilde{C}(I) = \tilde{C}(F)$, contradicting our hypothesis.

Therefore, assume that for at least one index i $(i = 1, 2, \ldots, k)$, $f_i(x)$ is not constant a.e. (P_F). Then there is a real number ζ such that for

$$S_\zeta = \{x : f_i(x) < \zeta, \quad x \in \mathscr{S}(F)\},$$

we have

$$0 < P_F(S_\zeta) < 1.$$

Clearly,

(1) $$f_i(S_\zeta) < f_i(S_\zeta^C \cap \mathscr{S}(F));$$

that is, for any $x \in S_\zeta$, $y \in S_\zeta^C \cap \mathscr{S}(F)$, we have $f_i(x) < f_i(y)$. Then we can construct a cumulative distribution function $F_1(x)$ with $\mathscr{S}(F_1) \subset S_\zeta$ such that there is a uniquely determined cumulative distribution function $F_2(x)$ with $\mathscr{S}(F_2) \subset (S_\zeta^C \cap \mathscr{S}(F))$, and

(2) $$F(x) = P_F(S_\zeta)F_1(x) + (1 - P_F(S_\zeta))F_2(x),$$

and by (1),

$$C_i(F_1) < C_i(F_2).$$

Thus by (2), we conclude that $\tilde{C}(F) \notin E(Q_{[a, b]})$, contradicting our hypothesis.

In particular, we have that $Q_{[a, b]}$ is the convex hull of $\tilde{C}(\mathscr{S}_{[a, b]})$, a result initially established in 1911 by F. Riesz [9].