DISCRETE APPROXIMATIONS OF DIFFERENTIAL OPERATORS BY SINC METHODS

by

Paul C. Gierke

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GRADUATE COLLEGE
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DISCRETE APPROXIMATIONS OF DIFFERENTIAL OPERATORS
BY SINC METHODS

Paul C. Gierke, Ph.D.
University of Nebraska, 1999

Advisor: Thomas S. Shores

This dissertation explores the discrete approximation of differential operators by Sinc methods. This leads into the areas of Sinc matrices and, more generally, Toeplitz matrices.

In the first part of the dissertation, we explore the Sinc matrices $I^{(n)}$ used to approximate derivative operators and their properties as a subset of skew-symmetric Toeplitz matrices. We will prove a key invertibility result for these Sinc matrices that covers all of the matrices, thereby definitively answering the open question of invertibility of the Sinc matrices $I^{(n)}$ for odd values of n.

In the second part of the dissertation, we seek to solve the first order system of equations

$$y' = f(x, y); \quad x \in \mathrm{R},\, y \in \mathrm{R}^n$$
$$y(a) = y_0$$

where f is a known vector-valued function in R^n. To accomplish this, we apply Sinc methods to discretize the problem. This generates a Toeplitz system of equations to solve with the special Sinc matrix $I^{(1)}$. In order to solve the system of equations given the unique properties of $I^{(1)}$ and matrices like it, we develop modified versions of three standard Toeplitz solvers that create a new class of hybrid routines set up to utilize the beautiful matrix/vector duality of the Toeplitz systems that these matrices produce.

The three categories of Toeplitz systems of equations that we address in this dissertation can be categorized as follows.

1. Yule-Walker equations $Ty = -r$: We solve $TY = -R$ with a modified version of the Durbin method. This system of equations arises in the solution of the two remaining classes of problems.
2. General right hand side equations $Tx = f$: We solve these with a modified version of the Levinson method.
3. Inversion of T: We solve this with a modified version of the Trench method.

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Contents

CHAPTER 1

Introduction

1.1. Introduction to the Two Matrix Problems

This dissertation explores the discrete approximation of differential operators by Sinc methods. This leads into the areas of Sinc matrices and, more generally, Toeplitz matrices. Toeplitz matrices see many applications in the solution of real-world problems and so are relatively well-studied in their symmetric form. However, many practical applications of Toeplitz matrices, including Sinc methods, may generate Toeplitz matrices that are skew-symmetric, of which little is known. A portion of this dissertation is devoted to Sinc matrices in general, in both their symmetric and skew-symmetric forms. However, the large portion of this dissertation is devoted to the study of the Sinc matrix $I^{(1)}$ used to approximate the first derivative operator and its properties as a subset of skew-symmetric Toeplitz matrices. We will also use this matrix to develop new results in approximating first order systems of ordinary differential equations. This necessitates a brief exploration of Sinc methods as well as an extended exposition of Toeplitz matrices and their applications in solving a series of three general problems.

1.2. Sinc Methods and Sinc Matrices

Sinc methods are a class of methods used to form a discrete approximation to a differential or integral operator. They generate Toeplitz matrices as discrete approximations to differential operators and are characterized by error estimates of the form $\mathcal{O}(e^{-c/h})$ where c is a constant and h can be considered to be a step size. This is far better than the usual polynomial error estimates in h, $\mathcal{O}(h^p)$, characteristic of other methods. Another advantage of Sinc methods is that they apply over a large variety of domains. They work well on the real line, semi-infinite intervals, and finite intervals, and are much more tolerant of singularities or near singularities near endpoints. In addition, they may be used on an arbitrary contour Γ in the complex plane.

In this dissertation, we utilize some of these strengths of Sinc methods as we apply them as a discrete approximation to a first order ordinary differential operator, but focus more on the Sinc matrix $I^{(1)}$ that arises in this approximation. We will also consider the Sinc matrices that arise as approximations to non-first order differential operators. The utility of these methods depends highly on the properties of these

matrices. These properties are critical in determining error bounds on solutions and, more importantly, in designing fast, efficient, accurate, and stable algorithms to implement Sinc methods. In particular, we develop a new way of defining the Sinc matrices by way of the existing $\delta_{jk}^{(n)}$ notation that lends itself to simple proofs of some known properties of these matrices. It also opens the door to future exploration of Sinc methods by way of difference equations. Next, we summarize and restate some of the few known properties of Sinc matrices, prove a key invertibility property, and, with this last proof in hand, improve and refine the new and preexisting matrix properties. This further adds to their utility in the area of Sinc methods.

1.3. Solving Toeplitz Systems of Equations

Toeplitz systems of linear equations arise in many disciplines of mathematics and engineering, such as time series analysis, image processing, control theory, statistics, integral equations, orthogonal polynomials, differential equations, and Padé approximations, to name a few. For the more extensive list, of which this is just a part, see [**1**]. By utilizing the symmetry inherent in the Toeplitz matrix, these systems can be solved with $\mathcal{O}(n^2)$ operations instead of the full $\mathcal{O}(n^3)$ operations typical of a solving a general system of linear equations. There is a plethora of routines for solving these systems in the many forms in which they appear. However, all of these routines trace their origins to three simple Toeplitz matrix problems with their own unique characteristics. We will follow the naming convention and classical problem development used in Golub and Van Loan [**4**] as we describe these three classes of problems below.

For the classical development of each of the three different Toeplitz problems presented below, we will share the same Toeplitz matrix described more fully in Section 1.5 but presented here for clarity.

DEFINITION 1.3.1. Let $\{r_1, r_2, \ldots, r_n\}$ be a sequence of (real) scalars. We define the Toeplitz matrix T as follows.

$$T = T_n = \begin{bmatrix} 1 & r_1 & \cdots & r_{n-2} & r_{n-1} \\ r_1 & 1 & \ddots & & r_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ r_{n-2} & & \ddots & 1 & r_1 \\ r_{n-1} & r_{n-2} & \cdots & r_1 & 1 \end{bmatrix}$$

For the classical methods, we require that T_n, as well as its principal submatrices

$$T_k = \begin{bmatrix} 1 & r_1 & \cdots & r_{k-2} & r_{k-1} \\ r_1 & 1 & \ddots & & r_{k-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ r_{k-2} & & \ddots & 1 & r_1 \\ r_{k-1} & r_{k-2} & \cdots & r_1 & 1 \end{bmatrix}$$

for $k = 1, 2, \ldots, n$, be positive definite (see Definition 1.5.13.) Note that by construction, they are symmetric (see Definition 1.5.2) as well. These will be the two key requirements of the classical methods that the Toeplitz matrices in this dissertation do not possess. Thus we are forced to develop the new methods described briefly below.

The Three Categories of Toeplitz Systems of Equations:

1. **Yule-Walker equations** $Ty = -r = -[r_1\ r_2 \cdots r_n]^T$**:**

   These systems arise in the solution of certain linear prediction problems as well as intermediate but integral steps in the solution process of the remaining two classes of Toeplitz matrix problems. Note also that the vector r is the special vector made up of the scalars $\{r_1, r_2, \ldots, r_n\}$ used to define the Toeplitz matrices T_k above. Classically, this is solved with a Durbin algorithm. The Durbin algorithm iteratively builds the solution vector y by first solving the system for $k = 1$ giving a solution y_1. The algorithm then uses that result to build the next step solution y_2 for the case $k = 2$. This one-step procedure continues until it reaches the n-th step solution $y = y_n$.

   The classical Durbin method is used for symmetric, positive definite Toeplitz matrices. In this dissertation, however, the Toeplitz matrix that we consider does not have either of these properties. In fact, for odd size Toeplitz matrices, the system we consider has no solution at all. This leads us to develop a hybrid method that in some sense takes a "two-step" approach to solving a slightly different system that has grown to the matrix equation $TY = -R$ from the

simpler, classical vector equation $Ty = -r$. That is, we solve the system

$$\begin{bmatrix} 0 & -1 & r_2 & \cdots & r_{2k-2} & r_{2k-1} \\ 1 & 0 & -1 & r_2 & \ddots & r_{2k-2} \\ -r_2 & 1 & 0 & \ddots & \ddots & \vdots \\ \vdots & -r_2 & \ddots & \ddots & -1 & r_2 \\ -r_{2k-2} & \ddots & \ddots & 1 & 0 & -1 \\ -r_{2k-1} & -r_{2k-2} & \cdots & -r_2 & 1 & 0 \end{bmatrix} \times$$

$$\begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \\ \vdots & \vdots \\ y_{2k-1,1} & y_{2k-1,2} \\ y_{2k,1} & y_{2k,2} \end{bmatrix} = - \begin{bmatrix} -1 & r_2 \\ r_2 & r_3 \\ r_3 & r_4 \\ \vdots & \vdots \\ r_{2k-1} & r_{2k} \\ r_{2k} & r_{2k+1} \end{bmatrix}$$

instead of the classical system

$$\begin{bmatrix} 1 & r_1 & \cdots & r_{k-2} & r_{k-1} \\ r_1 & 1 & \ddots & & r_{k-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ r_{k-2} & & \ddots & 1 & r_1 \\ r_{k-1} & r_{k-2} & \cdots & r_1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{k-1} \\ y_k \end{bmatrix} = - \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_{k-1} \\ r_k \end{bmatrix}.$$

To solve this new, more complicated Toeplitz system of equations, we develop a modified Durbin algorithm that shares the iterative component of its parent algorithm. Inherent in this modified algorithm are certain symmetry properties that we may exploit in generating computer code to numerically solve this system of equations. In particular, we will examine the symmetry of the Toeplitz matrix and the associated matrix R as well as use the constructed symmetry of the solution matrix Y to simplify the code necessary to implement this new solution method.

2. **General right hand side problem $Tx = f$:**

This is the same as the Yule-Walker equations except that no requirements are made on the right hand side vector f. That is, f may or may not have any relation to the scalars $\{r_1, r_2, \ldots, r_n\}$ used to define the Toeplitz matrices T_k above. Thus, this problem represents all general Toeplitz linear systems of equations, including the Yule-Walker equations as a special case. Classically, this problem is solved with a Levinson algorithm. The Levinson algorithm

iteratively builds the solution vector f by first solving the system for $k = 1$ giving a solution f_1. The algorithm then uses that result to build the next step solution f_2 for the case $k = 2$. This one-step procedure continues until it reaches the n-th step solution $f = f_n$. This mirrors the solution method of the Durbin algorithm.

In the development of the Levinson algorithm, the Yule-Walker equations exist as an intrinsic part of the solution process. In fact, to solve the general right hand side problem, one just solves the standard Yule-Walker equations with an extra set of "parallel" vector equations that generate f at each step. Thus, the Levinson algorithm depends on the Durbin algorithm as an essential subset of code.

As with the Yule-Walker equations above, this dissertation deals with a non-standard Toeplitz matrix which yields a slightly different problem to solve. However, even though the matrix T_k is different that the classical case, we may still solve a vector problem $Tx = f$ instead of being forced to solve a matrix problem as above. We must still develop a modified Levinson method though, since the Yule-Walker equations are intrinsic to this solution process. Thus, we must utilize the modified Durbin method in the generation of a modified Levinson method.

3. **Inversion of T:**

This problem, classical or not, is to generate the inverse of the Toeplitz matrix T_n for some value of n. As in the general right hand side problem, the solution process includes the solution of the Yule-Walker equations as an intrinsic component. The classical case was solved by Trench [**13**] with a slightly more general solution by Zohar [**16**]. Thus the solution method used in the classical case is called the Trench algorithm.

As with both problems above, this dissertation deals with a non-standard Toeplitz matrix which yields a slightly different inversion procedure. Thus we must develop a modified Trench method, since, as above, the Yule-Walker equations are intrinsic to this solution process. Thus, once again, we must utilize the modified Durbin method in the generation of a modified Trench method.

1.4. Solving the First Order Vector Initial Value Problem

Here we apply Sinc methods to approximate the solution of a first order system of equations.

$$y' = f(x, y); \quad x \in \mathrm{R},\ y \in \mathrm{R}^n \tag{1.4.1}$$

$$y(a) = y_0 \tag{1.4.2}$$

where f is a known vector-valued function in R^n. This system of equations is a first order vector initial value problem (IVP). If we take this problem in its simplest form, we may consider the example from [**11**] where we solve the problem on the whole real line. If y is a solution to (1.4.1) that satisfies $\lim_{x\to-\infty} = 0$ as well as the auxiliary condition $\lim_{x\to\infty} = 0$, then applying Sinc methods, we obtain a node approximation w_m to y by solving the Sinc system

$$\frac{1}{h} I_m^{(1)} w_m = -f_m(x_m, w_m) \tag{1.4.3}$$

where $I_m^{(1)}$ is the discrete Sinc approximation to the first derivative operator and $f_m(x_m, w_m)$ is the node evaluation of the original right hand side function f from (1.4.1).

The beauty of this system is that the Sinc matrix $I_m^{(1)}$ in (1.4.3) is a prime example of the Toeplitz matrices for which we develop the modified methods discussed in the previous section. Thus this Sinc approximation reduces to solving a linear system of equations using the modified Levinson method.

In more complex examples, or for problems on a semi-infinite or finite interval, this problem is made slightly more difficult by the addition of diagonal matrices to the approximation equation. In this case, we employ Newton's method to solve the nonlinear system generated by the discretization realized by Sinc methods.

This method could be extended quite naturally to boundary value problems (BVP's) if the function in question already satisfied the right hand boundary condition. One could also employ a shooting method or parallel shooting method using these Sinc approximations as the underlying algorithm to obtain accurate results on either a finite or infinite interval.

1.5. Linear Algebra

This section gives a brief review of some parts of linear algebra pertinent to our discussion. While much of this is contained in books on linear algebra and matrix theory, such as [**3**], [**2**], [**4**], [**6**], and [**8**], it is included here as a quick reference. It also presents a further exposition of certain less covered Toeplitz matrix properties critical to our discussion. Most of the standard facts about Toeplitz matrices (both finite and infinite) are contained in [**5**] and [**15**] and are based on the pioneering work

of Otto Toeplitz himself. An excellent reference by Toeplitz [**12**] ties these matrices to the Fourier coefficients of certain functions. This underlies many of the properties we present here as well as the new properties we will prove in Chapter 3.

DEFINITION 1.5.1. Let $A = [a_{ij}]$ be an $n \times n$ matrix. We define the *transpose* of A as $A^T = [a_{ji}]$ and the *conjugate transpose* of A as $A^* \equiv \overline{A}^T = [\overline{a}_{ji}]$.

DEFINITION 1.5.2. Let A be an $n \times n$ matrix. We say that A is *Hermitian* if $A^* = A$ and *skew-Hermitian* if $A^* = -A$. We say that the real matrix A is *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$. These matrices are a subset of the *normal matrices* for which $AA^* = A^*A$.

DEFINITION 1.5.3. We define the *exchange matrix* as the square matrix E with 1's down the counter diagonal (the diagonal from the lower left corner to the upper right corner of the matrix) and 0's elsewhere. The exchange matrix E is sometimes called the counteridentity.

PROPERTY 1.5.4. The exchange matrix E has the following properties:

1. $E^2 = I$
2. $E = E^T = E^{-1}$
3. Premultiplying a matrix by E reverses the rows of the matrix. This has the effect of inverting each column. For example,
$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 7 & 0 & 5 & -9 \\ -6 & -2 & 4 & 8 \\ 3 & 9 & -5 & -7 \end{bmatrix} = \begin{bmatrix} 3 & 9 & -5 & -7 \\ -6 & -2 & 4 & 8 \\ 7 & 0 & 5 & -9 \\ 1 & -3 & 2 & -1 \end{bmatrix}$$
4. Postmultiplying a matrix by E reverses the columns of the matrix. This has the effect of inverting each row. For example,
$$\begin{bmatrix} 1 & -3 & 2 & -1 \\ 7 & 0 & 5 & -9 \\ -6 & -2 & 4 & 8 \\ 3 & 9 & -5 & -7 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & -3 & 1 \\ -9 & 5 & 0 & 7 \\ 8 & 4 & -2 & -6 \\ -7 & -5 & 9 & 3 \end{bmatrix}$$

One notes that if $A = [a_{ij}]$ is an $n \times n$ matrix, then $EA = [a_{n+1-i,j}]$, $AE = [a_{i,n+1-j}]$, and $EAE = [a_{n+1-i,n+1-j}]$.

One set of matrices that can be defined using the exchange matrix E is the set of persymmetric matrices. Zohar gives a brief discussion of persymmetric matrices in [**16**], some of which is presented here for clarity.

DEFINITION 1.5.5. Let A be an $n \times n$ matrix. We say that A is *persymmetric* if $EA^TE = A$, i.e., $a_{ij} = a_{n+1-j,n+1-i}$. A persymmetric matrix is symmetric about its counter diagonal.

One key property of persymmetric matrices is that their inverses, when they exist, are persymmetric as well. This property is also shared by the symmetric, skew-symmetric, Hermitian, and skew-Hermitian matrices.

For an invertible persymmetric matrix A we see that this is true by examining the matrix identity

$$(A^T)^{-1}A^T = I.$$

If we recall that $EE = I$, we may write this equation as

$$(E(A^T)^{-1}E)(EA^TE) = I.$$

Now, since A is persymmetric, $EA^TE = A$, and this becomes

$$(E(A^T)^{-1}E)A = I.$$

Since matrix inverses are unique, this forces

$$A^{-1} = E(A^T)^{-1}E = E(A^{-1})^TE$$

as desired. Hence A^{-1} is also persymmetric.

Here we note some other useful equalities involving the persymmetric matrix A and the exchange matrix E. The first two equalities follow immediately from the definition of persymmetric. We state them and related equalities as a proposition.

PROPOSITION 1.5.6. *Let A be a persymmetric matrix and E the exchange matrix. Then the following equalities hold.*

$$EA = A^TE$$
$$AE = EA^T$$

If A is also invertible, we obtain the equalities

$$EA^{-1} = (A^{-1})^TE = (A^T)^{-1}E$$
$$A^{-1}E = E(A^{-1})^T = E(A^T)^{-1}$$

If A is symmetric (and invertible), these may be reduced to the two equalities

$$EA = AE$$
$$EA^{-1} = A^{-1}E$$

Finally, if A is skew-symmetric (and invertible) we obtain the two equalities

$$EA = -AE$$
$$EA^{-1} = -A^{-1}E$$

These equalities will be key to the development of the modified methods for dealing with Toeplitz systems of equations in that they allow a greater range of symmetry properties than do ordinary symmetric and skew-symmetric matrices.

DEFINITION 1.5.7. Let $T = [t_{ij}]$ be an $n \times n$ matrix. We say that T is a *Toeplitz matrix* if it is constant along each diagonal.

Stated another way, we say that an $n \times n$ matrix, $T = [t_{ij}]$, is a Toeplitz matrix if there exists a set of scalars

$$\{a_{-(n-1)}, a_{-(n-2)}, \ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots, a_{n-2}, a_{n-1}\}$$

such that $t_{ij} = a_{j-i}$ for all i and j. This yields the Toeplitz matrix

$$T = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & a_1 & a_2 & \ddots & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \ddots & \ddots & \vdots \\ \vdots & a_{-2} & \ddots & \ddots & a_1 & a_2 \\ a_{-(n-2)} & \ddots & \ddots & a_{-1} & a_0 & a_1 \\ a_{-(n-1)} & a_{-(n-2)} & \cdots & a_{-2} & a_{-1} & a_0 \end{bmatrix}$$

Thus we may completely specify an $n \times n$ Toeplitz matrix with any sequence of numbers $a_{-(n-1)}, \ldots, a_0, \ldots, a_{n-1}$.

PROPERTY 1.5.8. Toeplitz matrices are both normal and persymmetric.

For the purposes of this dissertation, we will also consider an alternate development of Toeplitz matrices that hearkens back to the original work by Otto Toeplitz. To aid us in this development, we must first state some pertinent definitions and properties of matrices and their eigenvalues, especially as they relate to the Sinc matrices of our dissertation.

DEFINITION 1.5.9. If A is an $n \times n$ matrix and v is a nonzero vector satisfying the equation

$$Av = \lambda v$$

for some number $\lambda \in C$, then the pair $\{\lambda, v\}$ is called an *eigenpair* of the matrix A. The scalar λ is called an *eigenvalue* of A and the vector v is called an *eigenvector* of A corresponding to the eigenvalue λ.

PROPERTY 1.5.10. The eigenvalues of a Hermitian matrix are real.

PROPERTY 1.5.11. The eigenvalues of a real symmetric matrix are real.

PROPERTY 1.5.12. The eigenvalues of a real skew-symmetric matrix are purely imaginary and come in complex conjugate pairs with the possible exception of $0i = 0$ which may occur as a single value.

DEFINITION 1.5.13. The complex Hermitian matrix A is *positive definite* if for all nonzero $v \in \mathrm{C}^n$,

$$v^* A v > 0.$$

DEFINITION 1.5.14. The real symmetric matrix A is *positive definite* if for all nonzero $v \in \mathrm{R}^n$,

$$v^* A v > 0.$$

PROPERTY 1.5.15 ([4] Corollary 4.2.2). If A is positive definite then all its principal submatrices are positive definite. In particular, all the diagonal entries are positive.

In order to prove other properties of Toeplitz matrices, especially results pertaining to eigenvalues, we consider another useful development of these matrices by way of finite Toeplitz forms.

DEFINITION 1.5.16. Let f be a real-valued function which is Lebesgue integrable on the interval $(-\pi, \pi)$, and let

$$t_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt, \quad n = 0, \pm 1, \pm 2, \ldots$$

be its Fourier coefficients. We define the *finite Toeplitz form* $T_n(f)$ by

$$\begin{aligned} T_n(f) &\equiv \vec{v}^* T_n \vec{v} \\ &= \sum_{j,k=0}^{n-1} t_{k-j} v_j \bar{v}_k \end{aligned}$$

where T_n is the Toeplitz matrix $T_n = [t_{jk}] = [t_{k-j}]$ and $\vec{v}$ is an arbitrary vector in C^n.

Since f is real-valued, we note that

$$\begin{aligned} \bar{t}_{-n} &= \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{int}\,dt} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \bar{f}(t)e^{-int}\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \\ &= t_n \end{aligned}$$

so that T_n is Hermitian and hence its eigenvalues are real.

If we substitute the formula for t_{k-j} in the definition of $T_n(f)$ above, we get the more useful version of the Toeplitz form, namely

$$\begin{aligned} T_n(f) &\equiv \vec{v}^* T_n \vec{v} \\ &= \sum_{j,k=0}^{n-1} t_{k-j} v_j \overline{v}_k \\ &= \sum_{j,k=0}^{n-1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-i(k-j)t}\, dt \right) v_j \overline{v}_k \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{j,k=0}^{n-1} v_j e^{ijt} \overline{v}_k e^{-ikt} \right) f(t)\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{p=0}^{n-1} v_p e^{ipt} \right|^2 f(t)\, dt. \end{aligned}$$

Thus if $f(t) \equiv 1$, we see that T_n is the $n \times n$ identity matrix I_n and we get that $T_n(f) = \sum_{p=0}^{n-1} |v_p|^2 = ||\vec{v}||_2^2$.

This last representation of the Toeplitz form leads to the following properties of the form and its associated matrix.

THEOREM 1.5.17 ([**5**] Section 5.2, [**14**]). *Let f be a real-valued function which is Lebesgue integrable on the interval $(-\pi, \pi)$. Denote the "essential" minimum and maximum of f by m and M respectively, so that $m \leq f(x) \leq M$ for all x in $(-\pi, \pi)$. If we denote the eigenvalues of T_n by $\{\lambda_j^{(n)}\}_{j=1}^n$, then $m \leq \lambda_j^{(n)} \leq M$ for $j = 1, 2, \ldots, n$. If we further assume that f is not constant on a set of measure 2π, then we get the strict inequality $m < \lambda_j^{(n)} < M$ for $j = 1, 2, \ldots, n$. Furthermore, if we order the eigenvalues of T_n by*

$$\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \cdots \leq \lambda_n^{(n)}$$

we also get the limiting property that

$$\lim_{n\to\infty} \lambda_1^{(n)} = m \qquad \text{and} \qquad \lim_{n\to\infty} \lambda_n^{(n)} = M.$$

DEFINITION 1.5.18. Let the square matrix M be given by

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A and D are square matrices, not necessarily of the same size. If A is nonsingular, set $S = D - CA^{-1}B$. We call S the *Schur complement* of A in M.

Using the Schur complement above we may write M as the block-LDU decomposition, called the Schur decomposition,

$$M = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}$$

from which it immediately follows that $\det M = \det A \cdot \det S$. Thus if M is nonsingular, then S is nonsingular as well, and we may invert the above decomposition to obtain the formula

$$\begin{aligned} M^{-1} &= \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ 0 & S^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \\ &= \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{bmatrix} \end{aligned}$$

DEFINITION 1.5.19. We define the *Hadamard product* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, denoted by $A \circ B$, to be the element-wise product of the two matrices $[a_{ij}b_{ij}]$.

DEFINITION 1.5.20. We define the *Kronecker product* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, denoted by $A \otimes B$, to be the product matrix $[a_{ij}B]$.

CHAPTER 2

Sinc Methods

2.1. Introduction to Sinc Methods

In this chapter we present the preliminary material necessary for a working understanding of Sinc methods as they are applied to differential equations. While Sinc methods may be applied to regions and arcs in the complex plane, we focus the discussion on their application to problems on the real line or a portion of it. Specifically, we consider how Sinc methods may be applied to problems on real domains that may be infinite, semi-infinite, or finite.

2.2. Functions and Domains

We begin by defining the well-known sinc function

$$\text{(2.2.1)} \qquad \operatorname{sinc}(z) = \begin{cases} \dfrac{\sin \pi z}{\pi z}, & z \neq 0, \\ 1, & z = 0. \end{cases}$$

While sinc is an entire function, we will often deal with $\operatorname{sinc}(x)$ for $x \in \mathrm{R}$, and the translated sinc function $S(j,h)$ defined by

$$\text{(2.2.2)} \qquad S(j,h)(x) = \operatorname{sinc}\left(\frac{x - jh}{h}\right), \qquad h > 0, \quad j \in \mathrm{Z}.$$

Thus we see that

$$\operatorname{sinc} x = S(0,1)(x).$$

One of the main interpolatory features of the sinc function is to note its values at the integer inputs x. We clearly see the values that sinc takes on at these points in Figure 1. Note that it takes on the value 1 at $x = 0$ and is 0 for every other integer value. The translated sinc function $S(j,h)$ behaves in a similar manner. It has a value of 1 at $x = jh$ and takes on the value 0 for all other integer multiples of h. Thus we see that $S(j,h)$ interpolates just as the standard sinc function but with a step size h.

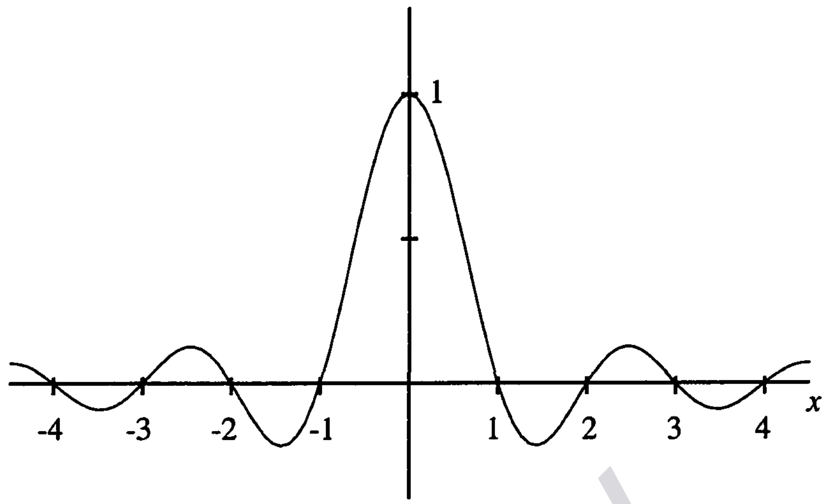

FIGURE 1. A graph of sinc x

One feature of Sinc methods is that they give exact interpolation and quadrature results for functions of the Paley-Wiener class, that is, functions that satisfy the following definition.

DEFINITION 2.2.1 ([9] Definition 2.2). Let h be a positive constant. The Paley-Wiener class of functions $B(h)$ is the family of entire functions f such that on the real line $f \in L^2(\mathrm{R})$ and in the complex plane f is of exponential type π/h, i.e.,

$$|f(z)| \leq K \exp(\pi|z|/h)$$

for some $K > 0$.

If we loosen the restriction that interpolation and quadrature are exact, we may still obtain results where the error decreases exponentially, for example, $\mathcal{O}(e^{-c/h})$ where c is a constant. This leads us to a definition of another slightly less restrictive class of functions.

DEFINITION 2.2.2 ([9] Definition 2.12). Let $\mathcal{D}_S$ denote the infinite strip domain (Figure 2) of width $2d$, $d > 0$, given by

$$\mathcal{D}_S \equiv \{w \in \mathrm{C} : w = u + iv, |v| < d\}. \tag{2.2.3}$$

Let $B^p(\mathcal{D}_S)$ be the set of functions analytic in $\mathcal{D}_S$ that satisfy

$$\int_{-d}^{d} |f(t+iv)|\, dv = \mathcal{O}(|t|^a), \quad t \to \pm\infty, \quad 0 \leq a < 1$$