OSCILLATION THEORY OF DYNAMIC EQUATIONS ON TIME SCALES

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Lynn H. Erbe and Allan C. Peterson

Lincoln, Nebraska

May, 2008

UMI Number: 3297856

UMI

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In past years mathematical models of natural occurrences were either entirely continuous or discrete. These models worked well for continuous behavior such as population growth and biological phenomena, and for discrete behavior such as applications of Newton's method and discretization of partial differential equations. However, these models are deficient when the behavior is sometimes continuous and sometimes discrete. The existence of both continuous and discrete behavior created the need for a different type of model. This is the concept behind dynamic equations on time scales. For example, dynamic equations can model insect populations that are continuous while in season, die out in, say, winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population.

Throughout this work, we will be concerned with certain dynamic equations on time scales. We start with a brief introduction to the time scale calculus and some theory necessary for the new results. The main concern will then be the oscillatory behavior of solutions to certain second order dynamic equations. In Chapter 3, an equation of particular interest is one containing both advanced and delayed arguments. We will use the method of Riccati substitution to prove some oscillation results of the solutions.

In Chapter 4 we again study the oscillatory behavior of a second dynamic equation. However, in this chapter, the equation only has delayed arguments. In addition to using Riccati substitution, we use the method of upper and lower solutions to develop necessary and sufficient conditions for oscillatory solutions. In the final chapter we are

interested in the existence of nonoscillatory solutions of dynamic equations on time scales. The common theme among these results is the use of the Riccati substitution technique and the integration of dynamic inequalities.

ACKNOWLEDGMENTS

I humbly take this opportunity to publicly thank all the individuals who have positively contributed to my education. I first give honor and praise to God. He has shown His omnipotence yet again.

I give my sincere appreciation to my parents, Reginald (Jacqueline) Higgins and Sharon (Gary) Reed. Their encouragement, love, and pride were an incredible source of support for me in this process. I am proud to be their daughter.

I am grateful to my family in the Mathematics Department at the University of Nebraska-Lincoln for endless support. In addition, I am thankful to my advisors, Dr. Lynn Erbe and Dr. Allan Peterson. While Dr. Peterson assisted me with the fine details of my work, Dr. Erbe helped me with the "big" picture. Their patience and confidence in my ability to succeed were remarkable. I would also like to thank my officemates for making this experience an unforgettable one.

I would have not considered graduate school if it were not for the encouragement of the mathematics faculty and staff of Xavier University of Louisiana. I would like to thank Dr. Vlajko Kocic for introducing me to research and the field of difference equations. I would also like to thank Mrs. Erica Houston for her thoughtful advice.

I am grateful to all the people that have entered my life and influenced me in countless ways. I will miss the love and support of the Mt. Zion Baptist Church family and the Lincoln Alumnae Chapter of Delta Sigma Theta Sorority, Inc. Thanks to the Black Graduate Student Association for teaching me how to embrace my heritage. Thanks to Tehia and April for making our house a home; love will forever dwell in 2806.

Lastly, I have to say *Asante Sana* to Kamau Oginga Siwatu. I am extremely blessed to have a mate that is loving, supportive, patient, and unselfish. Thanks for always being there.

Contents

Chapter 1

Introduction

The theory of time scales is a new area of mathematics that unifies and extends discrete and continuous analysis. The time scale calculus allows us to model situations in which the behavior is both continuous and discrete. For example, it can model insect populations that are continuous while in season, die out in, say, winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a non overlapping population.

In recent years there has been an increasing interest in studying the oscillation and nonoscillation of solutions of dynamic equations on time scales. Already many results concerning second order dynamic equations have been established [3, 7, 15]. In this present work we aim to extend the results of [11] and [16] to dynamic equations on time scales and to improve those of [26]. For oscillation of nonlinear delay dynamic equations, Zhang and Shanliang [26] considered the equation

$$y^{\Delta\Delta}(t) + q(t)f(y(t-\tau)) = 0, \quad t \in \mathbb{T} \tag{1.1}$$

where $\tau \in \mathbb{R}$ and $t-\tau \in \mathbb{T}$, $f : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing, and $uf(u) > 0$ for $u \neq 0$. By using comparison theorems, they proved that the oscillation of (1.1) is equivalent to that of the nonlinear dynamic equation

$$y^{\Delta\Delta}(t) + q(t)f(y^{\sigma}(t)) = 0, \quad t \in \mathbb{T} \tag{1.2}$$

where $\sigma(t)$ is the next point in the time scale, and established some sufficient conditions for oscillation by applying the results established in [9] for (1.2) on unbounded above time scales. In Chapter 3 we show that the oscillation of

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y(\tau(t))) = 0,$$

where $\tau(t)$ is a delay given by a function, τ, of t, is equivalent to that of

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0.$$

on an isolated time scale $\mathbb{T}$ where $\sup \mathbb{T} = \infty$.

In extending the results of [11] to dynamic equations on time scales, we establish oscillation criteria for the second order nonlinear dynamic equation

$$y^{\Delta\Delta} + f\left(t, y^{\sigma}(t), y(\tau(t))\right) = 0 \tag{1.3}$$

with retarded argument in Chapter 4. In order to obtain the results for (1.3), we improve and extend some results of [6] and [17].

In the final chapter, Chapter 5, we are interested in the asymptotic behavior of solutions of dynamic equations on time scales. In [16], the author obtains necessary and sufficient conditions for the existence of a bounded nonoscillatory solution of $y'' + f(t, y)g(y') = 0$ with a prescribed limit at ∞ and necessary and sufficient conditions for a nonoscillatory solution whose derivative has a positive limit at ∞. We extend some of these results to

$$y^{\Delta\Delta} + f(t, y^{\sigma})g(y^{\Delta}) = 0.$$

Chapter 2

Preliminaries

In this chapter we introduce some basic concepts concerning the calculus on time scales. Most of these results will be stated without proof. The proofs can be found in [2] and [5].

2.1 The Calculus on Time Scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Thus $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, i.e., the real numbers, the integers, the natural numbers, and the nonnegative integers are examples of time scales.

Definition 2.1.1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\},$$

and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\}.$$

In the case that $\{s \in \mathbb{T} : s > t\}$ is empty, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t). Similarly, if $\{s \in \mathbb{T} : s < t\}$ is empty, we put $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t).

If $f : \mathbb{T} \to \mathbb{R}$ is a function, we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}.$$

Points are classified as follows: If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$ we say t is *left-scattered*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is said to be *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. Points that are right-scattered and left-scattered at the same time are called *isolated*, and points that are both right and left dense are called *dense*.

Definition 2.1.2. The *graininess function*, $\mu : \mathbb{T} \to [0, \infty)$, is defined by

$$\mu(t) := \sigma(t) - t.$$

The *backward graininess function*, $\nu : \mathbb{T} \to [0, \infty)$, is defined by

$$\nu(t) := t - \rho(t).$$

Definition 2.1.3. We also need below the set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows: If the maximum, m, of $\mathbb{T}$ is left-scattered, then $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$.

Throughout this work we make the blanket assumption that a and b are points in $\mathbb{T}$. Often we assume $a \leq b$. We then define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

2.2 Differentiation

Now we consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the so-called delta derivative of f at a point $t \in \mathbb{T}^\kappa$. By convention we will define $\lim_{s \to t} f(s) = f(t)$ if t is an isolated point.

Definition 2.2.1. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s| \text{ for all } s \in U.$$

We call $f^\Delta(t)$ the *delta* (or *Hilger*) *derivative* of f at t. Moreover, we say that f is *delta differentiable* (or in short: *differentiable*) on $\mathbb{T}^\kappa$ provided f^Δ exists for all $t \in \mathbb{T}^\kappa$.

Some useful relationships concerning the delta derivative are now given.

Theorem 2.2.2. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are functions and let $t \in \mathbb{T}^\kappa$. Then we have the following:*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If t is right-dense, then f is differentiable at t iff the limit

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists as a finite number. In this case,

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

(iv) If f is differentiable at t, then $f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t)$.

Looking at properties *(ii)* and *(iii)* in the above theorem gives us a more intuitive understanding of the derivative that cannot be gained via the definition alone. If $t\in\mathbb{T}$ is right dense, then the delta-derivative behaves much the same way as the usual derivative. It can be viewed as the slope of the tangent line to the function at t, although if t is both right-dense and left-scattered, the limit is a one-sided limit. On the other hand, if t is right-scattered, then $f^{\Delta}(t)$ is the slope of the line segment containing $f(t)$ and $f(\sigma(t))$. In this instance, the behavior of the function to the left of t is irrelevant beyond the requirement that f is continuous at t. Thus, the delta derivative combines the discrete behavior of the forward difference operator and the continuous behavior of the usual derivative.

We next provide the theorem that allows us to find the derivative of sums, products, and quotients of differentiable functions.

Theorem 2.2.3. *Assume f, g are differentiable at* $\mathbb{T}^{\kappa}$. *Then:*

(i) The sum $f+g:f:\mathbb{T}\to\mathbb{R}$ *is differentiable at t with*

$$(f+g)^{\Delta}(t)=f^{\Delta}(t)+g^{\Delta}(t).$$

(ii) For any constant α, $\alpha f:\mathbb{T}\to\mathbb{R}$ *is differentiable at t with*

$$(\alpha f)^{\Delta}(t)=\alpha f^{\Delta}(t).$$

(iii) The product $fg:\mathbb{T}\to\mathbb{R}$ *is differentiable at t with*

$$(fg)^{\Delta}(t)=f^{\Delta}(t)g(t)+f(\sigma(t))g^{\Delta}(t)=f(t)g^{\Delta}(t)+f^{\Delta}(t)g(\sigma(t)).$$

(iv) If $g(t)g(\sigma(t))\neq 0$, *then* $\dfrac{f}{g}$ *is differentiable at t and*

$$\left(\frac{f}{g}\right)^{\Delta}(t)=\frac{f^{\Delta}(t)g(t)-f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Finally, we present a chain rule which calculates $(f \circ g)^{\Delta}$, where

$$g : \mathbb{T} \to \mathbb{R} \quad \text{and} \quad f : \mathbb{R} \to \mathbb{R}.$$

This chain rule is due to Christian Pötzsche, who derived it first in 1998.

Theorem 2.2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^{\kappa}$ and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t)$$

holds $t \in \mathbb{T}^{\kappa}$.

2.3 Integration

Of course, the calculus on time scales would not be complete without a concept of integration to complement the derivative. In order to describe functions that are "integrable," we introduce the following concept.

Definition 2.3.1. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted in this dissertation by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$C^1_{rd} = C^1_{rd}(\mathbb{T}) = C^1_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 2.3.2. A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^{\Delta}(t) = f(t) \quad \forall\, t \in \mathbb{T}^{\kappa}.$$

Then we define the *Cauchy integral* by

$$\int_a^b f(t)\Delta t = F(b) - F(a), \quad \forall\, a, b \in \mathbb{T}.$$

Theorem 2.3.3. *Every rd-continuous function has an antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^t f(\tau)\Delta\tau \quad \text{for} \quad t \in \mathbb{T}$$

is an antiderivative of f.

The following theorem provides useful properties of delta integrals.

Theorem 2.3.4. *If* $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R}$, *and* $f, g \in C_{rd}$, *then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t;$

(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t;$

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t;$

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t;$

(v) $\int_a^b f^\sigma(t)g^\Delta(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^\Delta(t)g(t)\Delta t$ *where* f, g *can be interchanged;*

(vi) If $|f(t)| \leq g(t)$ *on* $[a, b)$, *then*

$$\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t;$$

(vii) if $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\Delta t \geq 0$.

The following result provides useful properties of the delta integral.

Theorem 2.3.5. *Let* $a, b \in \mathbb{T}$ *and* $f \in C_{rd}$.

(i) If $\mathbb{T} = \mathbb{R}$, *then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) If $[a, b]$ *consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if } a < b \\ 0 & \text{if } a = b \\ -\sum_{t\in[b,a)} \mu(t)f(t) & \text{if } a > b. \end{cases}$$

Chapter 3

Oscillation Criteria for Functional Dynamic Equations

3.1 Oscillation of Nonlinear Dynamic Equations

We shall consider the second order nonlinear functional dynamic equation

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y(\tau(t))) = 0 \tag{3.1}$$

and the second order nonlinear dynamic equation

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0 \tag{3.2}$$

on an isolated time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$. We assume $p, q, \tau,$ and f satisfy the following **Condition (*E*)**:

(i) $p \in C_{rd}(\mathbb{T}, (0, \infty))$ satisfies $\displaystyle\int_{t_0}^{\infty} \frac{1}{p(t)}\,\Delta t = \infty, \quad t \in \mathbb{T}.$

(ii) $q \in C_{rd}(\mathbb{T}, \mathbb{R}^{+})$.

(iii) $\tau \in C_{rd}(\mathbb{T}, \mathbb{T})$ satisfies

$$\lim_{t\to\infty} \tau(t) = \infty \quad \text{and} \quad \exists\, M > 0 \text{ such that } |R(t) - R(\tau(t))| < M \;\forall\, t \in \mathbb{T}$$

where $R(t) = \displaystyle\int_{t_0}^{t} \frac{1}{p(s)}\,\Delta s.$

(iv) $f : \mathbb{R} \to \mathbb{R}$ is continuous, increasing, and

$$f(-u) = -f(u) \text{ for } u \in \mathbb{R} \quad \text{and} \quad uf(u) > 0 \text{ for } u \neq 0.$$

By a solution of (3.1) we mean a nontrivial real-valued function y satisfying (3.1) for $t \geq t_0 \geq a \in \mathbb{T}$, where $a > 0$. A solution y of (3.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (3.1) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of $(py^{\Delta})^{\Delta} + q(t)f(y(t)) = 0$ which exist on some half line $[t_y, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|y(t)| : t > t_0\} > 0$ for any $t_0 \geq t_y$.

Definition 3.1.1. A nonempty closed subset K on a Banach space X is called a cone if it possess the following properties:

(i) if $\alpha \in \mathbb{R}^+$ and $x \in K$, then $\alpha x \in K$.

(ii) if $x, y \in K$, then $x + y \in K$.

(iii) if $x \in K \setminus \{0\}$, then $-x \in K$.

Let X be a Banach space and K be a cone with nonempty interior. Then we define a partial ordering $\leq$ on X by

$$x \leq y \quad \text{if and only if} \quad y - x \in K.$$

Our main result, which follows, is an extension of Theorem 2.1 of [26].

Theorem 3.1.2. *Assume (E) holds and* $\dfrac{\mu(t)}{p(t)}$ *is bounded. We further assume* $\tau(t) \leq \sigma(t)$ *for all* t *or* $\tau(t) \geq \sigma(t)$ *for all* t*. Then the oscillation of the second order nonlinear dynamic equation*

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0 \tag{3.2}$$

is equivalent to the oscillation of the second order nonlinear functional dynamic equation

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y(\tau(t))) = 0. \tag{3.1}$$

We will need the following fixed-point theorem [10].

Theorem 3.1.3. *(Knaster's Fixed-Point Theorem) Let* X *be a partially ordered Banach space with ordering* $\leq$*. Let* Ω *be a subset of* X *with the following properties: The infimum of* Ω *belongs to* Ω *and every nonempty subset of* Ω *has a supremum which belongs to* Ω*. If* $S : \Omega \to \Omega$ *is an increasing mapping, then* S *has a fixed point in* Ω*.*

In order to prove Theorem 3.1.2, we will need to begin with the following lemmas.

Lemma 3.1.4. *Assume that (E) holds. A necessary and sufficient condition for equation (3.2) to be oscillatory is that, the inequality*

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) \leq 0, \tag{3.3}$$

has no eventually positive solutions.

Proof. SUFFICIENCY. Assume (3.3) has no eventually positive solutions. Then neither does (3.2), and so $(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0$ is oscillatory. If y is an eventually negative solution of (3.2), then let $x = -y$. Then x is eventually positive and

$$(px^{\Delta})^{\Delta} + qf(x^{\sigma}) = -(py^{\Delta})^{\Delta} - qf(y^{\sigma}) = -\left[(px^{\Delta})^{\Delta} + qf(x^{\sigma})\right] = 0$$

for $t \geq T$ sufficiently large by Condition (E) (iv). Thus x is an eventually positive solution of (3.3), which is a contradiction. Hence, $(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0$ is oscillatory.

NECESSITY. Suppose that (3.2) is oscillatory, and by way of contradiction, assume that (3.3) has an eventually positive solution y, namely, there exists $t_0 \in \mathbb{T}$ ($t_0 \geq a$) such that $y(t) > 0$ for $t \geq t_0$. As $\sigma(t) \geq t$ for all t, $\sigma(t) \geq t_0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then $y^{\sigma}(t) > 0$ for $t \geq t_0$. Using this fact along with the sign condition on f in (E), we have $[p(t)y^{\Delta}(t)]^{\Delta} \leq 0$ for $t \geq t_0$, and so $p(t)y^{\Delta}(t)$ is decreasing on $[t_0, \infty)_{\mathbb{T}}$.

We claim that $y^{\Delta}(t) > 0$ for all large t. If not, then for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$, we have $y^{\Delta}(t_1) \leq 0$. It follows that $p(t)y^{\Delta}(t) \leq 0$, $t \in [t_1, \infty)$. Now, if $y^{\Delta}(t_2) < 0$ for some $t_2 \geq t_1$, then

$$\begin{aligned} y(t) - y(t_2) &= \int_{t_2}^{t} y^{\Delta}(s)\,\Delta s \\ &= \int_{t_2}^{t} \frac{p(s)y^{\Delta}(s)}{p(s)}\,\Delta s \\ &\leq p(t_2)y^{\Delta}(t_2)\int_{t_2}^{t} \frac{\Delta s}{p(s)} \\ &\to -\infty \text{ as t} \to \infty, \end{aligned}$$

which is a contradiction to our assumption that $y(t) > 0$ for $t \geq t_0$. Hence it follows that $y^{\Delta}(t) \equiv 0$ on $[t_1, \infty)$, and so $(p(t)y^{\Delta}(t))^{\Delta} \equiv 0$ and $q(t)f(y^{\sigma}(t)) > 0$, which is contradictory. Consequently, there exists $T \in \mathbb{T}$ ($T \geq t_0$) such that

$$y(t) > 0, \qquad y^{\Delta}(t) > 0, \qquad \text{and} \qquad (p(t)y^{\Delta}(t))^{\Delta} \leq 0$$

for all $t \geq T$. Since $p(t)y^{\Delta}(t)$ is continuous, the integrals below are well-defined. Integrating $(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) \leq 0$ from t to s yields

$$p(s)y^{\Delta}(s) - p(t)y^{\Delta}(t) + \int_{t}^{s} q(u)f(y^{\sigma}(u))\,\Delta u \leq 0, \quad \text{for} \quad s, t \in \mathbb{T} \quad \text{and} \quad s \geq t,$$

i.e.,

$$p(t)y^{\Delta}(t) \geq p(s)y^{\Delta}(s) + \int_{t}^{s} q(u)f(y^{\sigma}(u))\,\Delta u. \tag{3.4}$$

Since $p(t)y^{\Delta}(t) > 0$ is decreasing for $t \geq T$, $\lim_{t\to\infty} p(t)y^{\Delta}(t) = k \geq 0$ exists. Letting $s \to \infty$ in (3.4) we obtain

$$y^{\Delta}(t) \geq \frac{k}{p(t)} + \frac{1}{p(t)} \int_t^{\infty} q(u)f(y^{\sigma}(u))\,\Delta u \geq \frac{1}{p(t)} \int_t^{\infty} q(u)f(y^{\sigma}(u))\,\Delta u. \tag{3.5}$$

Since $\int_t^{\infty} q(u)f(y^{\sigma}(u))\,\Delta u$ exists and is continuous, integrating (3.5) from T to t yields

$$y(t) \geq y(T) + \int_T^t \frac{1}{p(s)} \int_s^{\infty} q(u)f(y^{\sigma}(u))\,\Delta u\,\Delta s, \qquad t \geq T. \tag{3.6}$$

Define X to be the Banach space of all continuous functions on $[a,\infty)_{\mathbb{T}}$ satisfying $\lim_{t\to\infty} x(t) = \infty$, where $\|\cdot\|$ is defined by

$$\|x\| := \max_{t\in[a,\infty)_{\mathbb{T}}} |x(t)| \quad \text{for all} \quad x \in X.$$

Let

$$\Omega := \left\{\omega \in C([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+) : 0 \leq \omega(t) \leq 1 \text{ and } \lim_{t\to\infty} \omega(t) = \infty \text{ for } t \geq t_0\right\},$$

which is endowed with the usual pointwise ordering $\leq$: $\omega_1 \leq \omega_2 \Leftrightarrow \omega_1(t) \leq \omega_2(t)$ for $t \geq t_0$.

One can show that for any nonempty subset N of Ω $\sup N \in \Omega$ and $\inf \Omega \in \Omega$. Define a mapping S on Ω by

$$(S\omega)(t) = \begin{cases} 1, & \text{if } t_0 \leq t \leq T, \\ \frac{1}{y(t)} \left(y(T) + \int_T^t \frac{1}{p(s)} \int_s^{\infty} q(u)f(y^{\sigma}(u)\omega^{\sigma}(u))\,\Delta u\,\Delta s\right), & \text{if } t \geq T. \end{cases}$$

We claim that $S\Omega \subset \Omega$ and S is monotone increasing. For any $\omega \in \Omega$, $(S\omega)(t)$ is certainly continuous and for $t \geq T$,

$$q(t)f(y^{\sigma}(t)\omega^{\sigma}(t)) \leq q(t)f(y^{\sigma}(t))$$

since $0 \leq \omega^{\sigma}(t) \leq 1$ and f is nondecreasing. Therefore, from (3.6), it follows that $0 \leq (S\omega)(t) \leq 1$ for $t \geq T$, and so $S(\omega) \in \Omega$. Moreover, if $\omega_1 \leq \omega_2$, $\omega_1, \omega_2 \in \Omega$, then, since f is nondecreasing, $f(y^{\sigma}(u)\omega_1(u)) \leq f(y^{\sigma}(u)\omega_2(u))$ and so $(S\omega_1)(t) \leq (S\omega_2)(t)$. Therefore, by Knaster's Fixed Point Theorem, there exists $\tilde{\omega} \in \Omega$ such that $S\tilde{\omega} = \tilde{\omega}$. Hence,

$$\tilde{\omega}(t) = \frac{1}{y(t)} \left(y(T) + \int_T^t \frac{1}{p(u)} \int_u^{\infty} q(v)f(y^{\sigma}(v)\tilde{\omega}^{\sigma}(v))\,\Delta v\,\Delta u\right), \text{ for } t \geq T.$$