PRIME IDEALS IN LOW-DIMENSIONAL MIXED POLYNOMIAL/POWER SERIES RINGS

by

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PRIME IDEALS IN LOW-DIMENSIONAL MIXED POLYNOMIAL/POWER SERIES RINGS

Christina Eubanks-Turner, Ph. D.

University of Nebraska, 2008

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Let B be a simple birational extension of $R[[x]]$, the ring of power series in one variable over a one-dimensional Noetherian domain R. We consider Spec(B), the set of prime ideals of B partially ordered by inclusion. The main result of Chapter 2 is a characterization of Spec(B), where R is a countable PID or an order in an algebraic number field. Among other results in Chapter 3 is a characterization of Spec$(R[[x]][y]/\mathbf{Q})$ for certain height-one prime ideals $\mathbf{Q}$ in $R[[x]][y]$, where R is a countable semilocal one-dimensional Noetherian domain. In Chapter 4 we give some properties of prime spectra of certain three-dimensional mixed polynomial/power series rings.

DEDICATION

Trust in the Lord with all your heart, and lean not on your own understanding; in all ways acknowledge Him, and He shall direct your paths. (Proverbs 3:5-6 NKJV)

I dedicate this work to my savior Jesus Christ. This dissertation is a testimony of his love and faithfulness to me. I also want to dedicate this work to my two Precious Gifts from God, Amari and Earl. I love both of you dearly and I truly thank both of you for your support. Lastly, I want to dedicate this to my wonderful, beautiful, nurturing mother. Thank you for your love and encouragement.

ACKNOWLEDGMENTS

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Contents

Chapter 1

Introduction

For a commutative Noetherian ring R with identity, we consider the *prime spectrum* of R, denoted Spec(R). This is the set of prime ideals of R, a partially-ordered set (poset) under inclusion. In 1950 Irving Kaplansky asked: "What partially-ordered sets occur as the prime spectrum of a Noetherian ring?" This question remains open. A related question was answered by Melvin Hochster in 1969; he described those topological spaces that are homeomorphic to Spec(R) with the Zariski topology for some commutative ring R [10]. W. James Lewis and Jack Ohm showed that every finite partially-ordered set occurs as Spec(R), for some commutative ring R (not necessarily Noetherian) [12]. In 1976 Roger Wiegand and Sylvia Wiegand [24] determined the partially-ordered sets that occur as j-Spec(R), that is, those prime ideals that are intersections of maximal ideals, for R some countable Noetherian ring. Kaplansky's question has also been investigated by William Heinzer, Ray Heitmann, Steve McAdam, Jack Ratliff, Roger Wiegand and Sylvia Wiegand in [6, 4, 5, 8, 9, 15, 16, 22, 23].

Kaplansky's question is still open even when restricted to Noetherian domains of dimension less than or equal to two. For certain two-dimensional rings, the prime spectrum has been characterized. In 1986, Roger Wiegand [23] gave an axiomatic

characterization of the prime spectrum of the polynomial ring $\mathbb{Z}[y]$, where $\mathbb{Z}$ is the ring of integers and y is an indeterminate. He showed that the prime spectrum of every two-dimensional domain that is finitely generated as a k-algebra, where k is a subfield of the algebraic closure of a finite field, is isomorphic to the prime spectrum of $\mathbb{Z}[y]$, but the prime spectrum of the ring $\mathbb{Q}[x, y]$, where $\mathbb{Q}$ is the ring of rational numbers, is not isomorphic to $\mathrm{Spec}(\mathbb{Z}[y])$. Also in [23], R. Wiegand showed that, for D an order in an algebraic number field, $\mathrm{Spec}(D[y]) \cong \mathrm{Spec}(\mathbb{Z}[y])$. He also conjectured:

Every two-dimensional domain that is a finitely generated $\mathbb{Z}$-algebra has spectrum order-isomorphic to $\mathbb{Z}[y]$.

A *birational extension* B of R is a Noetherian integral domain between R and the quotient field of R. In 1996 Aihua Li and Sylvia Wiegand proved that Roger Wiegand's conjecture holds for every finitely generated birational extension B of $\mathbb{Z}[y]$, cf. [13]. Precisely, they showed that $\mathrm{Spec}(\mathbb{Z}[y]) \cong \mathrm{Spec}(B)$, whenever $B := \mathbb{Z}[y][\frac{g_1}{f}, ..., \frac{g_m}{f}]$, such that $f, g_1,..,g_m \in \mathbb{Z}[y]$, and f is nonzero. Some additional progress on proving the conjecture was made by Serpil Saydam and Sylvia Wiegand in 1998: The conjecture holds for every finitely generated extension of $D[y]$ contained in $K[y]$, where K is the quotient field of D, and D is an order in an algebraic number field [19].

My goal is to work on Roger Wiegand's conjecture and to classify prime spectra of various polynomial/power series rings. In 2006, William Heinzer, Christel Rotthaus, and Sylvia Wiegand determined the prime spectrum of $R[[x]]$, where R is a one-dimensional Noetherian domain [3]. The spectrum of the ring $R[[x]]$ is not order-

isomorphic to that of $\mathrm{Spec}(k[[x]][y])$, where k is a field and x, y are indeterminates. The spectrum of $k[[x]][y]$ differs from the spectrum of $k[y][[x]]$ and these prime spectra are different from $\mathrm{Spec}(\mathbb{Z}[x])$ and $\mathrm{Spec}(\mathbb{Q}[x, y])$.

In Chapter 2, an axiomatic characterization of the prime spectra of certain power series rings is given. This is joint work with Melissa Luckas and Serpil Saydam. In particular, we have a classification of the prime spectra of certain birational extensions of $R[[x]]$, where R is a countable principal ideal domain or an order in an algebraic number field.

In 1989, William Heinzer and Sylvia Wiegand characterized the prime spectrum of $R[y]$, for R a countable one-dimensional semilocal domain [4]. Then in 1996 Shah extended Heinzer and S.Wiegand's characterization of $R[y]$ to arbitrary cardinalities [21]. In Chapter 3, a characterization of the prime spectrum of $R[[x]][y]/\mathbf{Q}$ is given, where R is a countable semilocal domain of dimension one and $\mathbf{Q}$ is a height-one prime ideal of $R[[x]][y]$, with $(\mathbf{Q}, y) \neq 1$. In addition, other prime spectra of three-dimensional mixed polynomial/power series rings modulo a height-one prime are classified.

All of the rings mentioned above (except $R[[x]][y]$) have Krull dimension two. In Chapter 4 some axioms are given to describe prime spectra of certain three-dimensional Noetherian rings, such as partial characterizations for $\mathrm{Spec}(R\llbracket x\rrbracket[y])$, $\mathrm{Spec}(R[y]\llbracket x\rrbracket)$, $\mathrm{Spec}(R\llbracket x, y\rrbracket)$, where $(R, \mathbf{m})$ is a countable, local Noetherian domain of dimension one.

Throughout assume every ring R is a commutative Noetherian ring with 1.

Chapter 2

Prime Ideals in Birational Extensions of Two-Dimensional Power Series Rings

2.1 Introduction

It is convenient to set some notation for discussing partially ordered sets.

Notation 2.1.1. Let U be a partially-ordered set and $T \subseteq U$. For each $u \in U$, set

$$G_U(u) := \{w \in U \,|\, u < w\},\ L_U(u) := \{w \in U \,|\, w < u\},\ L_e(T) := \{x \in U \mid G_U(x) = T\}$$

(we say $G(u)$ or $L(u)$ if U is understood). For $u, v \in U, G(u, v) = G(u) \cap G(v)$.

Let R be a commutative ring. For $U = \operatorname{Spec}(R)$ and $\mathbf{p} \in U$, we define $G(\mathbf{p}) := \{\mathbf{q} \in U \mid \mathbf{p} \subset \mathbf{q}\}$.

Definition 2.1.2. A Noetherian ring R is a *catenary* if every saturated chain joining two prime ideals $\mathbf{p}$ and $\mathbf{p}'$, where $\mathbf{p} \subset \mathbf{p}'$ has maximal length height $\mathbf{p}'/\mathbf{p}$. (A chain is $\mathbf{p} \subset \mathbf{p}_1 \ldots \subset \mathbf{p}_{n-1} \subset \mathbf{p}'$ is saturated if no additional prime ideals can be inserted)

A *birational* extension of an integral domain R is a ring between R and its quotient

field. A finitely generated birational extension of R is a ring $B := R[\frac{g_1}{f}, \ldots, \frac{g_m}{f}]$, where $f, g_1, \ldots, g_m \in R$ and f is nonzero.

Definition 2.1.3. Let U be a two-dimensional partially ordered set with infinitely many height-two elements. Suppose that

1) The set $\mathcal{S} = \{u \in U \mid \mathrm{ht}(u) = 1,\ |G(u)| = \infty\}$ is finite set; say $\mathcal{S} = \{u_1, \ldots, u_n\}$.
2) $\displaystyle\bigcup_{i=1}^{n} G(u_i) = \{\text{height-two maximal elements of } U\}$.
3) $|G(u_i, u_j)| < \infty$, for every $i \neq j$, $1 \leq i, j \leq n$.

Then $\mathcal{S}$ is called the *set of special* elements of U and $u_1, \ldots, u_n$ are called *special* elements of U.

For a ring R, j-Spec(R) = {prime ideals of R that are intersections of maximal ideals}. The elements of j-Spec(R) are called j-primes.

> **Remark 2.1.4.** In Definition 2.1.3, if $U = \mathrm{Spec}(R)$ for some ring R, we refer to the elements $u_1, \ldots, u_n$ as the special primes of R.

Recall from [6] and [7] some remarks about j-primes of certain two-dimensional Noetherian rings.

Remarks 2.1.5. [6, 7] Let R be a Noetherian domain of dimension two with infinitely many height-two maximal ideals and such that $\mathcal{S} = \{u \in U \mid \mathrm{ht}(u) = 1,\ |G(u)| = \infty\}$ is finite.

1) A nonmaximal j-prime is the intersection of infinitely many maximal ideals and it is the intersection of every infinite set of maximal ideals that contain it.

2) A height-one prime ideal $\mathbf{P}$ of R is special if and only if $\mathbf{P}$ is a nonmaximal j-primes.

3) The two-dimensional poset Spec(R) is determined by the maximal ideals of R and the containments between the non-maximal height-one prime ideals and the height-two maximal ideals. In order to determine Spec(R) we consider two things: (i) j-Spec(R)

(ii) $L_e(T)$, where T ranges over finite sets of height-two maximal ideals of R.

The next list of axioms, given in [3], describe $U := \mathrm{Spec}(R[[x]])$ for R a one-dimensional Noetherian domain. The seven axioms in (a) are valid whenever R is a one-dimensional Noetherian domain. When R is countable, every two posets satisfying the seven axioms are order-isomorphic to each other (see b) and c) below).

Proposition 2.1.6. [3, Theorem 3.4] Let R be a one-dimensional Noetherian domain. Let x be an indeterminate over R. Let $\beta := |R[[x]]|$ and let $\alpha := \big|\{\text{maximal ideals of } R\}\big|$. Set $U := \mathrm{Spec}(R[[x]])$. Then

(a) U with the partial order inclusion satisfies the following axioms:

(I) $|U| \leq \beta$, where $\beta = |R|^{\aleph_0}, \aleph_0 = |\mathbb{N}|$.

(II) U has a unique minimal element, namely (0).

(III) $\dim(U) = 2$ and $|\{\text{height-two elements of } U\}| = \alpha$.

(IV) There exists a unique special height-one element $u_1 \in U$ (namely, $u_1 = (x)$) and $u_1 < t$, for every height-two element t of U.

(V) For every nonspecial height-one element u of U there exists exactly one height-two element t with $u < t$.

(VI) For every height-two element $t \in U$, $L_e(\{t\})$ is uncountable. For t_1, t_2 distinct height-two elements of U, if u is a height-one element of U with $u \in L(t_1) \cap L(t_2)$, then $u = u_1$.

(VII) Every maximal element has height two.

(b) If R is countable, then $U = \mathrm{Spec}(R[[x]])$ satisfies (II)-(V), (VII) and the stronger axioms (I′) and (VI′):

(I′) $|U| = \beta$.

(VI′) For every height-two element $t \in U$, $|L_e(\{t\})| = \beta$.

(c) Every partially-ordered set satisfying axioms (I′), (II)-(V), (VI′) and (VII) is order-isomorphic to every other such partially-ordered set.

Definition 2.1.7. A partially-ordered set U is *Henselian affine of type* $(\beta, \alpha)_2$ if U satisfies (I)-(VII). The subscript $_2$ indicates that every maximal element has height two.

With the hypothesis of Proposition 2.1.6 the prime spectrum of $R[[x]]$ looks like this:

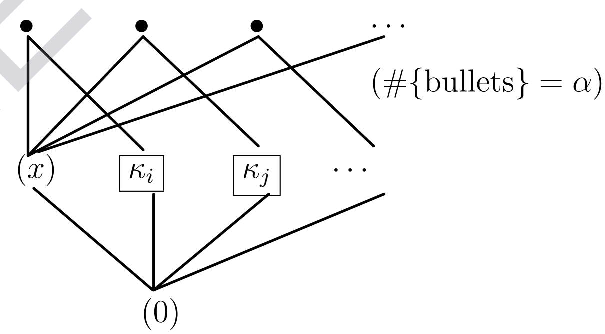

Diagram 2.1.7.1

Here α is the cardinality of the set of maximal ideals of R; the boxed κ_i (one for each maximal ideal of R) means that there are κ_i prime ideals in that position, where each κ_i is uncountable and $\kappa_i \leq \beta$. If R is countable, then $\kappa_i = c$ for all i, where $c = |\mathbb{R}|$. In this case every two posets described by Diagram 2.1.7.1 are order-isomorphic.

Next we give a list of axioms describing $\text{Spec}(k[[x]][y])$, for k a field and x, y indeterminates over k.

Proposition 2.1.8. [3, Theorem 3.1] Let k be a field and set $U := \text{Spec}(k[[x]][y])$. Let $\alpha := |k[y]|$ and $\beta := |k[[x]]|$. Then U is characterized by the following axioms:

(1) $|U| = \beta$.

(2) U has a unique minimal element.

(3) $\dim(U) = 2$ and $|\{\text{height-two elements of U}\}| = \alpha$.

(4) There exists a unique special height-one element $u_1 \in U$ (namely, $u_1 = (x)$) and $u_1 < t$, for every height-two element t of U.

(5) For every nonspecial height-one element u of U, there exists at most one height-two element t with $u < t$.

(6) For every height-two element $t \in U$, $|L_e(\{t\})| = \beta$. For t_1, t_2 distinct height-two elements of U, if u is a height-one element of U with $u \in L(t_1) \cap L(t_2)$, then $u = u_1$.

(7) $|\{\text{height-one maximal elements of } U\}| = \beta$.

Note that axiom (4) of Proposition 2.1.8 implies that u_1 is contained in every height-two element.

Definition 2.1.9. A partially-ordered set U is *Henselian affine of type* (β, α) if U satisfies (1)-(7).

The prime spectrum picture of $k[[x]][y]$ is shown below.

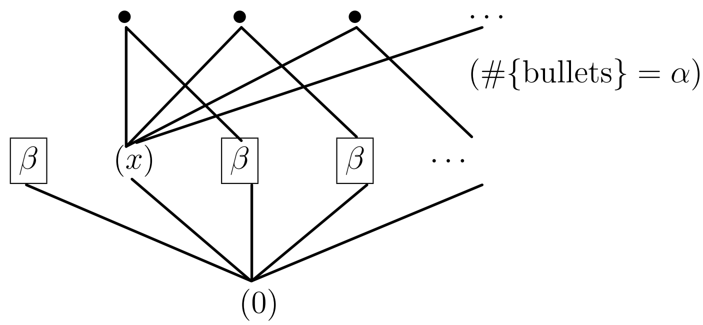

Diagram 2.1.9.1

From the diagrams and axioms, one can see that the two partially-ordered sets described in Proposition 2.1.6 and Proposition 2.1.8 are quite similar. One major difference between the axioms of Propositions 2.1.6 and 2.1.8 is that the partially-ordered set in Proposition 2.1.8 has height-one maximal elements while the other has none.

Remark 2.1.10. 1) For R an integral domain, $\mathrm{Spec}(R)$ has a unique minimal element. This holds throughout since we work exclusively with integral domains.

2) Every one-dimensional domain R is Cohen-Macaulay. Therefore $R[[x_1,\ldots,x_n]][y_1,\ldots,y_m]$ and $R[y_1,\ldots,y_m][[x_1,\ldots,x_n]]$ are Cohen-Macaulay, where x_i, y_j are indeterminates over R, for all i, j with $1 \le i \le n, 1 \le j \le m$, cf. [14, Theorem 17.7]. Since each of $R[[x_1,\ldots,x_n]][y_1,\ldots,y_m]$ and $R[y_1,\ldots,y_m][[x_1,\ldots,x_n]]$ is Cohen-Macaulay, Theorem 17.9 in [14] implies each of these rings is catenary.

The following proposition shows that removing finitely many nonspecial height-one elements and their greater than set from the partially-ordered set described in Proposition 2.1.6 gives the partially-ordered set described in Proposition 2.1.8.

Proposition 2.1.11. Let X be a Henselian affine partially ordered set of type $(\beta,\alpha)_2$ for infinite cardinals α and β. Let $x \in X$ be the special height-one element from (IV)

of Proposition 2.1.6, and let $u_1, ..., u_n$ be height-one elements of X such that $x \neq u_i$, for every i, $1 \leq i \leq n$. Let $U := X \setminus (\{u_1, ..., u_n\} \cup (\bigcup_{i=1}^{n} G_X(u_i)))$. Then U is Henselian affine of type (β, α).

Proof. To show U is Henselian affine of type (β, α), we show that axioms (1)-(7) of Proposition 2.1.8 are satisfied. For (1), notice that, for each i, $|G_X(u_i)| = 1$ by (V) since each $u_i \neq x$, for all i with $1 \leq i \leq n$. Therefore $|\bigcup_{i=1}^{n} G_X(u_i)| \leq n$. Since $|X| = \beta$ is infinite, $|U| = |X| - |\bigcup_{i=1}^{n} G_X(u_i)| - n = \beta$.

For (2), the minimal element of X is the unique minimal element of U.
Since $U \subseteq X$, $\dim(U) \leq 2$. Now $\dim(U) = 2$ since removing only finitely many elements of height one and height two leaves infinitely many elements of height one and height two by (III) and (VI) and [11, Theorem 88].

The rest of (3) and axioms (4), (5), and the first part of (6) hold by the construction of U from X; only finitely many height-one and height-two elements are removed. The second part of (6) holds for U, since it holds in X.

To show (7), for each i, $1 \leq i \leq n$, consider the height-two element $G_X(u_i)$. In X, $G_X(u_i)$ contains β height-one elements of X, each exactly less than $G_X(u_i)$ by (VI). In U, for all i, $1 \leq i \leq n$, the height-one elements exactly less than $G_X(u_i)$ in X, other than u_i itself, become height-one maximal elements of U. Thus by (VI), there are β height-one maximal elements in U. Thus (1)-(7) of Proposition 2.1.8 hold and the statement holds. □

Removing finitely many nonspecial height-one elements and their greater than sets from a poset that is Henselian affine of type (β, α) does not change the isomorphism class of the poset, as the next proposition shows.

Proposition 2.1.12. Let X be a partially-ordered set that is Henselian affine of type

(β, α) for infinite cardinals α and β. Let $x \in X$ be the special height-one element from item 4 of Proposition 2.1.8, and let $u_1, ..., u_n$ be height-one elements of X with $x \neq u_i$, for every i, $1 \leq i \leq n$. Let $U = X \setminus (\{u_1, ..., u_n\} \cup (\bigcup_{i=1}^{n} G_X(u_i)))$. Then U is Henselian affine of type (β, α).

Proof. Since X is Henselian affine of type (β, α) and $x \neq u_i$, for every i, $1 \leq i \leq n$, $\left|\{u_1, ..., u_n\} \cup (\bigcup_{i=1}^{n} G_X(u_i))\right|$ is finite. Therefore U remains Henselian affine of type (β, α).□

In the current work, we consider some two-dimensional rings that have prime spectra similar to a Henselian affine spectrum. However, the spectra of these rings have special elements of three types. A partially-ordered set with more than one special element is described below. This is a generalization of the definition in [6].

Definition 2.1.13. Let m and n be nonnegative integers, not both 0 and let α, β be infinite cardinals. We call a partially-ordered set U is *j-birational of type* (m, n, α, β) provided U satisfies the following axioms:

(1) $\beta := |\{\text{height-one maximal elements}\}|$.

(2) U has a unique minimal element.

(3) $\dim(U) = 2$ and $|\{\text{height-two elements of U}\}| = \alpha$.

(4) U has exactly $n + m + 1$ nonmaximal special height-one elements: $H_1, \ldots, H_m$, $Q_1, \ldots, Q_n$, and P, and the following hold:

i) $|G_U(H_i)| = |G_U(Q_j)| = |G_U(P)| = \alpha$ for all i, j, $1 \leq i \leq m$ and $1 \leq j \leq n$.

ii) $|G_U(Q_j, P)| = 1$ for all $j, 1 \leq j \leq n$.

iii) Every remaining pair of special elements is comaximal.

Diagram 2.1.13.1 is a picture of a j-birational partially ordered set of type (m, n, α, β).

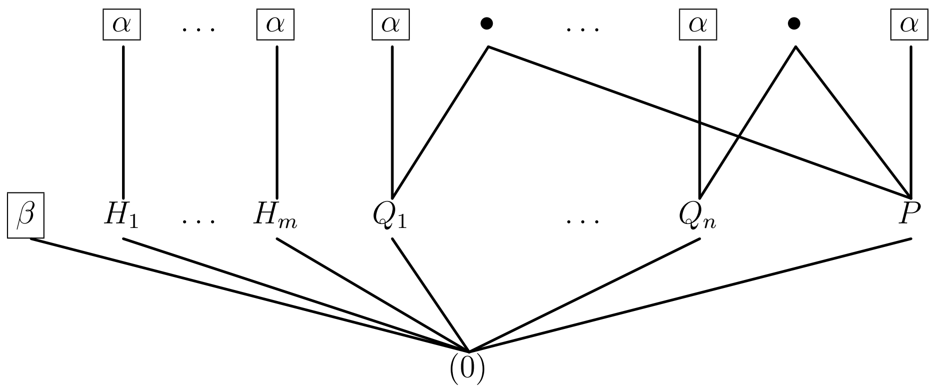

Diagram 2.1.13.1

Proposition 2.1.14. Let m and n be nonnegative integers, not both 0, and let α, β be infinite cardinals. If U and V are two posets that are both j-birational of type (m, n, α, β), then $U \cong V$.

Proof. We define an order-preserving bijection γ from U to V. Define $\gamma(u_0) := v_0$, where u_0 is the unique minimal element of U and v_0 is the unique minimal element of V. Since U and V are both j-birational of type (m, n, α, β), for i, j, $1 \leq i \leq n$ and $1 \leq j \leq m$, let the u_i, u'_j, u and the v_i, v'_j, v be the special elements of U and V, respectively such that $|G_U(u_j, u)| = 1 = |G_V(v_j, v)|$, for every j, $1 \leq j \leq n$. Then define γ by u_i maps to v_i, u'_j to v'_j, and u to v, for all i, $1 \leq i \leq m$ and j, $1 \leq j \leq n$. Since $|G_U(u_i)| = \alpha = |G_V(v_i)|$ and u_i, v_i are comaximal with every other special element of U and V, respectively, we have a bijection from $G_U(u_i)$ to $G_V(v_i)$, for each i, $1 \leq i \leq n$ and we define γ there to be that bijection. Since $|G_U(u'_j, u)| = 1 = |G_V(v'_j, v)|$, we define $\gamma(G_U(u'_j) - G_U(u'_j, u)) = G_V(v'_j) - G_V(v'_j, v)$, for each j, $1 \leq j \leq n$. Let $\{\mathbf{m}_1, \ldots, \mathbf{m}_n\} = \bigcup_{j=1}^{n} G_U(u, u'_j)$ and let $\{\mathbf{m}'_1, \ldots, \mathbf{m}'_n\} = \bigcup_{j=1}^{n} G_V(v, v'_j)$. Define $\gamma(\mathbf{m}_j) = \mathbf{m}'_j$ for each $j, 1 \leq j \leq n$ and $\gamma\big(G_U(u) - \bigcup_{j=1}^{n} G_U(u, u'_j)\big) = G_V(v) - \bigcup_{j=1}^{n} G_V(v, v'_j)$. Thus we have a mapping from the height-two elements of U to the height-two elements of V. Now map the β height-one maximal elements of U to the β height-one maximal

elements of V via a set bijection. This defines an order-preserving isomorphism from U to V. $\square$

Given a birational extension B, we further classify j-primes of B as *survivors* or *transient* following [6].

Definition 2.1.15. [6] For a birational extension $B := R[[x]][\frac{g_1}{f}, \ldots, \frac{g_m}{f}]$, where $f, g_1, \ldots, g_m \in R[[x]]$ and f is a nonzero nonunit, the nonmaximal height-one j-primes $\mathbf{p}$ of B that *survive* in $B[\frac{1}{f}]$, that is, $\mathbf{p}B[\frac{1}{f}] \neq B[\frac{1}{f}]$, are called *survivors*. Those that do not survive, that is, $\mathbf{p}B[\frac{1}{f}] = B[\frac{1}{f}]$, are called *transient*.

Examples of rings having j-spectra that are j-birational of type (m, n, α, β), are given in Proposition 2.1.8 for $m = 1, n = 0$. The next proposition is a result of Heinzer, Lantz and S. Wiegand. (the nonnegative integers m and n are determined by a factorization of f, g over $k[x]$).

Proposition 2.1.16. [6, Theorem 2.6] Let $(R, \mathbf{m}, k)$ be a local domain of dimension one, $\{f, g\}$ an $R[x]$-sequence, $f \notin \mathbf{m}[x]$ and $B := R[x][\frac{g}{f}]$. Then j-Spec(B) is j-birational of type (m, n, α, α), where $\alpha = |R[x]|$. Furthermore, the special primes of B other than $\mathbf{p}$ are all transient, and $\mathbf{p} = \mathbf{m}B[\frac{1}{f}] \cap B$ is a survivor.

We define another partially-ordered set that has the same axioms (1) through (3) as Definition 2.1.13. This poset is simpler than the j-birational poset because here the special elements are pairwise comaximal.

Definition 2.1.17. Let $r \geq 1$ and let α, β be infinite cardinals. A partially-ordered set U is *j-Henselian affine of type* (r, α, β) provided U satisfies the following axioms:

(1) $|\{\text{height-one maximal elements}\}| = \beta$.

(2) U has a unique minimal element.

(3) $\dim(U) = 2$ and $|\{\text{height-two elements of U}\}| = \alpha$.

(4) U has exactly r nonmaximal special height-one elements $H_1, \ldots, H_r$, and the following hold:

(i) $|G_U(H_i)| = \alpha$, for all i, $1 \leq i \leq r$.

(ii) $G_U(H_i) \cap G_U(H_j) = \emptyset$, for $i \neq j$.

A picture of a j-Henselian partially ordered set is given in Diagram 2.1.17.1

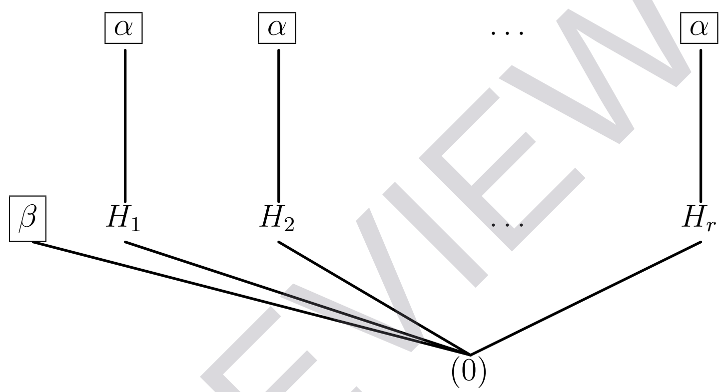

Diagram 2.1.17.1

Proposition 2.1.18. Let $r \geq 1$ and let α, β be infinite cardinals. If U and V are two posets that are both j-Henselian affine of type (r, α, β), then $U \cong V$.

Proof. The proof is similar to the proof of Proposition 2.1.14. □

2.2 General facts about Noetherian rings

Notation 2.2.1. Let R be a ring and let $r \in R$.
Define $V_R(r) := \{\mathbf{p} \in \mathrm{Spec}(R) \,|\, r \in \mathbf{p}\}$.

We use the following general fact about localizations from [13].

Proposition 2.2.2. [13, Proposition 3.1] Suppose that C is an integral domain, $0 \neq c \in C$, and D is a birational extension such that $C \subseteq D \subseteq C[\frac{1}{c}]$. Let $\mathbf{p}_0 \in \mathrm{Spec}(D) \setminus V_D(c)$. Then there exists a prime ideal $\mathbf{Q} \in \mathrm{Spec}(C) \setminus V_C(c)$ such that $\mathbf{p}_0 = \mathbf{Q}C[\frac{1}{c}] \cap D$. Furthermore, for every $\mathbf{Q} \in \mathrm{Spec}(C) \setminus V_C(c)$, $\mathbf{p}_0 = \mathbf{Q}C[\frac{1}{c}] \cap D$ is a prime ideal of $D \setminus V_D(c)$, and $\mathbf{p}_0 \cap C = \mathbf{Q}$.

The following result generalizes Remark 2.3.2 in [13].

Proposition 2.2.3. Let R be an e-dimensional Noetherian domain, and let $x, y_1, ..., y_n$ be indeterminates over R. Each height $n+e+1$ maximal ideal of $R[[x]][y_1, ..., y_n]$ has the form $(\mathbf{m}, x)R[[x]][y_1, ..., y_n]$, where $\mathbf{m}$ is a maximal ideal of $R[y_1, ..., y_n]$.

Proof. Let $\mathbf{p}$ be a maximal ideal of $R[[x]][y_1, ..., y_n]$ of height $n+e+1$. Let $\mathbf{m} := \mathbf{p} \cap R[y_1, ..., y_n]$, a prime ideal of $R[y_1, ..., y_n]$. We claim that $(\mathbf{m}, x) = \mathbf{p}$. Clearly $(\mathbf{m}, x) \subseteq \mathbf{p}$, since $x \in \mathbf{p}$ by [3, Remark 2.3.2]. Now let $f \in \mathbf{p}$. Then f is a polynomial in the variables $y_1, \ldots, y_n$ with coefficients in $R[[x]]$, and so $f = xs + t$, where $t \in R[y_1, ..., y_n]$ and $s \in R[[x]][y_1, ..., y_n]$. Also $t \in \mathbf{p}$, since $t = f - xs$ and both f and x are in $\mathbf{p}$. Since t is in $R[y_1, ..., y_n]$, we have $t \in \mathbf{m}$. Thus $f = xs+t \in (\mathbf{m}, x)$. To see that $\mathbf{m}$ is a maximal ideal of $R[y_1, ..., y_n]$, suppose that $\mathbf{q} \subseteq R[y_1, ..., y_n]$ is an ideal with $\mathbf{m} \subsetneq \mathbf{q}$. Then there exists an $a \in (\mathbf{q} \setminus \mathbf{m}) \cap R[y_1, ..., y_n]$, and so $a \in (\mathbf{q}, x) \setminus (\mathbf{m}, x)$. Thus $(\mathbf{m}, x) \subset (\mathbf{q}, x)$ and, since $(\mathbf{m}, x)$ is maximal in $R[[x]][y_1, ..., y_n]$, $(\mathbf{q}, x)$ is the unit ideal. Now $1 = \sum_{i=1}^{m} \alpha_i f_i + gx$, where $\alpha_i \in \mathbf{q}$ and $f_i, g \in R[[x]][y_1, ..., y_n]$. Let $x = 0$ and write out f_i as $f_i(x, y_1, ..., y_n)$; we have $1 = \sum_{i=1}^{m} \alpha_i f_i(0, y_1, ..., y_n) + 0 \in \mathbf{q}$, since $\alpha_i \in \mathbf{q}$, and $f_i(0, y_1, ..., y_n) \in R[y_1, ..., y_n]$. Thus $1 \in \mathbf{q}$, and so $\mathbf{q}$ is not a proper ideal, and so the result holds. □

Definition 2.2.4. Let R be a Noetherian domain and let $a, b \in R$ with a, b nonzero nonunits. Then $\{a, b\}$ is a *generalized R-sequence* if one of the following holds:

(1) $(a,b)R = R$, or

(2) $\{a,b\}$ is an R-sequence.

Proposition 2.2.5. [11, (9,p.102, Example 3)] Let S be a domain, let y be an indeterminate over S and let $s,t \in S$ be nonzero nonunits. If $\{s,t\}$ is a generalized S-sequence, then $(s+ty)$ is a prime ideal of $S[y]$.

Remark 2.2.6. Let R be a one-dimensional Noetherian domain, and let $B := R[[x]][\frac{g}{f}]$, where $\{f,g\}$ is a generalized $R[[x]]$-sequence and f and g are nonzero nonunits of $R[[x]]$.

1) Let $\pi : R[[x]][y] \longrightarrow \dfrac{R[[x]][y]}{(fy-g)}$ be the canonical homomorphism. We shall frequently use the following fact: There is a one-to-one order-preserving correspondence between the prime ideals $\mathbf{P}$ of $R[[x]][y]$ which contain $(fy-g)$ and the prime ideals $\mathbf{p}$ of $\dfrac{R[[x]][y]}{(fy-g)}$ given by $\mathbf{P} = \pi^{-1}(\mathbf{p})$ and $\mathbf{p} = \pi(\mathbf{P})$, written $\overline{\mathbf{P}} = \mathbf{p}$. Since $B \cong \dfrac{R[[x]][y]}{(fy-g)}$, we henceforth identify the prime ideals of B with the prime ideals of $R[[x]][y]$ that contain $(fy-g)$, so that each $\mathbf{P} \in \operatorname{Spec}(R[[x]][y])$ corresponds to $\pi(\mathbf{P}) = \overline{\mathbf{P}} \in \operatorname{Spec}(B)$.

2) Since $R[[x]][y]$ is catenary, the correspondence above also implies that height n prime ideals of B can be identified with height $n+1$ prime ideals of $R[[x]][y]$ containing $(fy-g)$.

3) By the correspondence in 1) and Proposition 2.2.3, the set of height-two maximal ideals of B is in one-to-one correspondence to the set of maximal ideals of $R[[x]][y]$ containing $(fy-g, x)$.

Proposition 2.2.2 does not directly yield a one-to-one correspondence of prime ideals in $R[\frac{1}{d}][[x]][\frac{g}{f}]$ and $R[[x]][\frac{g}{f}]$, where $0 \neq d \in R$, because $R[\frac{1}{d}][[x]][\frac{g}{f}]$ is not the

same as $R[[x]][\frac{g}{f}][\frac{1}{d}]$, cf. [S]. Thus we revise Proposition 2.2.2 in Lemma 2.2.7 to obtain a correspondence between some of the prime ideals of the rings $R[\frac{1}{d}][[x]][\frac{g}{f}]$ and those of $R[[x]][\frac{g}{f}]$.

Lemma 2.2.7. *Let R be a one-dimensional Noetherian domain and let x and y be indeterminates. Assume $\{f, g\}$ is a generalized $R[[x]]$-sequence. Let $d \in R$ be such that $\dim(R[\frac{1}{d}]) = 1$ and $d \notin (fy - g, x)$ in $R[[x]][y]$. Then there is a one-to-one correspondence between the height-two maximal ideals of $B := R[\frac{1}{d}][[x]][\frac{g}{f}]$ and the height-two maximal ideals of $R[[x]][\frac{g}{f}]$ that do not contain d, so that for every $\mathfrak{n}$ maximal in B, $\mathfrak{n} \cap B$ is maximal.*

Proof. By Remark 2.2.6, there is a one-to-one correspondence between the height-two maximal ideals of B and the height-three maximal ideals of $R[\frac{1}{d}][[x]][y]$ containing $(fy-g)$. Proposition 2.2.3 implies that the height-three maximal ideals of $R[\frac{1}{d}][[x]][y]$ have the form $(\mathbf{m}, x)$, where $\mathbf{m}$ is a maximal ideal of $R[\frac{1}{d}][y]$. Since $R[\frac{1}{d}][y] = R[y][\frac{1}{d}]$, $\mathrm{Spec}(R[\frac{1}{d}][y]) \cong \mathrm{Spec}(R[y]) - V_{R[y]}(d)$ by Proposition 2.2.2. Thus the height-three maximal ideals of $R[\frac{1}{d}][[x]][y]$ have the form $(\mathbf{m}, x)$, where $\mathbf{m}$ is a maximal ideal of $R[y]$ that does not contain d. Since we are considering only the maximal ideals of $R[[x]][y]$ that contain $(fy - g)$, we have the result. $\square$

Proposition 2.2.8. Let e, n be nonnegative integers with $n + e \geq 2$, let R be a catenary e-dimensional Noetherian domain, and let $\mathbf{Q}$ and $\mathbf{P}$ be prime ideals of $R[[x]][y_1 \ldots y_n]$, with $x \notin \mathbf{Q}$, $(\mathbf{Q}, x) \neq (1)$ and $(\mathbf{Q}, x)$ is not maximal. Also assume that $\mathbf{P}$ is a height $n + e + 1$ maximal ideal containing $\mathbf{Q}$ and that $\mathrm{ht}(\mathbf{Q}) = n + e - 1$. Then $G(\mathbf{Q}) \cap L(\mathbf{P})$ has uncountably many height $n + e$ prime ideals.

Proof. By Proposition 2.2.3, $\mathbf{P} = (\mathbf{m}, x)$, where $\mathbf{m}$ is a maximal ideal of $R[y_1 \ldots y_n]$, of height $n + e$. Now $\mathbf{Q}$ is a height $n + e - 1$ ideal of $R[[x]][y_1 \ldots y_n]$ with $x \notin \mathbf{Q}$ and

$\mathbf{Q} \subseteq (\mathbf{m}, x)$. Let $N_1, ..., N_m$ be the minimal primes of $(\mathbf{Q}, x)$ with $N_i \subseteq (\mathbf{m}, x)$ for all i, $1 \leq i \leq m$. Since $\mathbf{Q}$ is a height $n+e-1$ prime ideal and $\mathbf{Q} \subset (\mathbf{Q}, x)$, Krull's principal ideal theorem and the catenary condition imply $\mathrm{ht}(N_i) = n+e$, for all i, $1 \leq i \leq m$. Note that, since $x \notin \mathbf{m}$ and $N_1, ..., N_m$ are minimal primes of $(\mathbf{Q}, x)$, $\mathbf{m} \nsubseteq N_1 \cup ... \cup N_m$ by prime avoidance. We choose $a \in \mathbf{m} \setminus (N_1 \cup ... \cup N_m)$. Let $H := \big\{a + \sum_{i=1}^{\infty} w_i x^i \big| w_i \in \{a, 1\}\big\}$. Each of the elements of H is in $(\mathbf{m}, x)$, and $|H| = 2^{\aleph_0}$ is uncountable.

Claim: Let $\mathbf{p}$ be a height $n+e$ prime ideal of $R[[x]][y_1 \dots y_n]$ with $\mathbf{Q} \subseteq \mathbf{p} \subseteq (\mathbf{m}, x)$. Then $\mathbf{p}$ contains at most one element from the set H.

Proof of Claim: For the sake of contradiction, suppose $\mathbf{p}$ contains $h_1 = a + \sum_{i=1}^{\infty} w_i x^i$ and $h_2 = a + \sum_{i=1}^{\infty} v_i x^i$, where $h_1 \neq h_2$. Then

$$h_1 - h_2 = \sum_{i=1}^{\infty} w_i x^i - \sum_{i=1}^{\infty} v_i x^i = \sum_{i=1}^{\infty} (w_i - v_i) x^i \in \mathbf{p}.$$

Now let t be the smallest integer so that $w_t \neq v_t$. Then $h_1 - h_2 = x^t((w_t - v_t) + (w_{t+1} - v_{t+1})x + ...) \in \mathbf{p}$, and, as $\mathbf{p}$ is prime, $x \in \mathbf{p}$ or $(w_t - v_t) + (w_{t+1} - v_{t+1})x + ... \in \mathbf{p}$. If $x \in \mathbf{p}$, then $(\mathbf{Q}, x) \subseteq \mathbf{p} \subseteq (\mathbf{m}, x)$, and so, since $\mathrm{ht}(\mathbf{p}) = n+e$, we have $\mathbf{p} = N_i$, for some i. Since h_1 and $x \in \mathbf{p}$, we get $a \in \mathbf{p}$, contradicting the choice of a. If $(w_t - v_t) + (w_{t+1} - v_{t+1})x + ... \in \mathbf{p}$, then $(w_t - v_t) + (w_{t+1} - v_{t+1})x + ... \in (\mathbf{m}, x)$ and so $w_t - v_t \in (\mathbf{m}, x)$. But $w_t - v_t = \pm(1-a) \notin (\mathbf{m}, x)$ because $a \in \mathbf{m}$, a contradiction.

Thus, for each $h \in H$, there is a height $n+e$ prime ideal contained in $\mathbf{P} = (\mathbf{m}, x)$ and containing the ideal $(\mathbf{Q}, h)$. Since H is an uncountable set, there are uncountably many height $n+e$ prime ideals in $G(\mathbf{Q}) \cap L((\mathbf{P}))$. $\square$

Remark 2.2.9. Let $\mathbf{Q} \in \mathrm{Spec}(R[[x]][y_1 \dots y_n])$ with $x \notin \mathbf{Q}$ and $\mathrm{ht}(\mathbf{Q}) = n+e-1$. Assume either $(\mathbf{Q}, x) = (1)$ or $(\mathbf{Q}, x)$ is maximal. By Proposition 2.2.3, x is in every