SET-VALUED EXTENSIONS OF FUZZY LOGIC:

CLASSIFICATION THEOREMS

GILBERT ORNELAS

Department of Computer Science

APPROVED:

Vladik Kreinovich, Chair, Ph.D.

Luc Longpré, Ph.D.

Mohamed Amine Khamsi, Ph.D.

Pablo Arenaz, Ph.D.
Dean of the Graduate School

SET-VALUED EXTENSIONS OF FUZZY LOGIC:

CLASSIFICATION THEOREMS

by

GILBERT ORNELAS

THESIS

Presented to the Faculty of the Graduate School of

The University of Texas at El Paso

in Partial Fulfillment

of the Requirements

for the Degree of

MASTER OF SCIENCE

Department of Computer Science

THE UNIVERSITY OF TEXAS AT EL PASO

December 2007

UMI Number: 1449742

UMI Microform 1449742

ProQuest Information and Learning Company
300 North Zeeb Road
P.O. Box 1346
Ann Arbor, MI 48106-1346

Acknowledgements

I would like to express my most sincere thanks to all the people who made this possible. First and foremost I want to thank my parents for supporting and advising me on every decision that I have made throughout my life. I would not be writing this without them.

I wish to thank Dr. Vladik Kreinovich for being my advisor during this process and having the patience to work with me. He has always been very available and willing to help even when he is very busy or is already helping other people. I appreciate the fact that he always has a simple life example for every point he wants to make – this really helped in the understanding of concepts. I thank him for forcing me to be extremely clear when expressing my ideas or thoughts. From him I take a lot of lessons that can be applied not only to school and research but to life in general. Thanks.

I would like to thank the member of my committee, Dr. Luc Longpré and Dr. Amine Khamsi, for their help and for their patience.

I would also like to thank Dr. Ann Gates. She has always believed in me – even at times when I have doubted myself. I thank her for the opportunity she gave me as a research assistant. I learned a great deal while I was part of her research group. She is a very important role model to me.

Finally, I would like to thank two very special people – Leonardo Salayandia and Mary Contreras. More than co-workers, they became very good friends who have been there for me not only during the good times, but also to provide constructive criticism and advice during the not-so-good times. Their work ethic is just unbelievable, and I value their friendship very much.

NOTE: This thesis was submitted to my Supervising Committee on November, 2007.

Abstract

Experts are often not 100% confident in their statements. One of the most widely used approaches to describe the different degrees of confidence is the approach of *fuzzy logic*. In traditional fuzzy logic, the expert's degree of confidence in each of his or her statements is described by a number from the interval $[0, 1]$. However, due to similar uncertainty, an expert often cannot describe his or her degree by a *single* number. It is therefore reasonable to describe this degree by, e.g., a *set* of numbers. In this thesis, we show that under reasonable conditions, the class of such sets coincides:

- either with the class of all 1-point sets (corresponding to the traditional fuzzy logic),
- or with the class of all subintervals of the interval $[0, 1]$ (corresponding to the *interval-valued* fuzzy logic),
- or with the class of *all* closed subsets of the interval $[0, 1]$.

Thus, if we want to go beyond the traditional fuzzy logic and still avoid sets of arbitrary complexity, we have to use intervals. This classification result shows the importance of interval-valued fuzzy logics.

Table of Contents

Chapter 1

Introduction

1.1 Need for Fuzzy Logic

It is necessary to describe expert knowledge inside the computer. In many knowledge areas, such as geology, medicine, we rely on experts to make decisions. In each such area, there are a few top experts who have more knowledge than others and who, therefore, make the best decisions. It is desirable to make their expert knowledge available to other experts, so that they can use the expertise of these top experts in their own decision making.

Thus, we need to represent expert knowledge in a computer in such a way that the computer will be able to perform logical reasoning based on this knowledge.

Knowledge representation: successes. Since the early days of computing, there have been computer systems for representing expert knowledge. Probably the most well known idea behind such a representation is the use of Prolog, a special programming language designed for representing knowledge and for reasoning based on this knowledge.

It is necessary to take into account expert uncertainty. Prolog-type systems are a very good tool for describing expert statements which are absolutely correct. For example, if we have a statement that every person participating in a triathlon can swim and we assume this statement to be absolutely correct, then a Prolog-type system would be appropriate to represent this statement and combine it with other absolutely correct statements through reasoning rules to make further deductions.

However, it is not always the case that we have absolutely correct (or absolutely incorrect) statements to base our reasoning on. Instead, statements may be neither absolutely correct nor absolutely incorrect. One such example would be an expert meteorologist predicting that a hurricane will strike a certain area tomorrow morning. This statement introduces uncertainty, since the meteorologist is not absolutely certain that this area will be affected by the hurricane.

In order to adequately represent expert knowledge, it is necessary to take the expert's uncertainty into account.

Fuzzy logic: a formalism for representing expert uncertainty. In a Prolog-type system, every statement is either true or false. In the computer, "true" is usually represented as 1, and "false" is usually represented as 0. The value "true" corresponds to absolute certainty, and value "false" corresponds to the absolute lack of certainty. To represent intermediate degrees of certainty, it is therefore reasonable to use numbers intermediate between 0 and 1: the larger the number, the more we are certain about the given statement.

This idea of using numbers from the interval $[0,1]$ to describe the experts' uncertainty is called a *fuzzy logic*; see, e.g., [2, 5]. Fuzzy logic was first introduced by L. A. Zadeh exactly for this purpose – to formally describe this uncertainty in human reasoning. In fuzzy logic, a person's degree of certainty is described by a number from the interval $[0,1]$, so that absolute certainty in a statement corresponds to 1, absolute certainty in its negation corresponds to 0, and intermediate values, or pure uncertainty, correspond to intermediate degrees of certainty.

1.2 Composite Statements in Fuzzy Logic

Need for composite statements. Our objective is not simply to represent the uncertainty, but also to process this uncertainty.

For example, if we have two statements S_a and S_b, and we know the degree of certainty a in the statement S_a and the degree of certainty b in the statement S_b, then we need to estimate the degree of certainty in a composite statement like $S_a \wedge S_b$ or $S_a \vee S_b$.

Need for an "and" operation. The desired estimate for the degree of certainty of $S_a \wedge S_b$ must depend on the degrees a of S_a and b of S_b. In other words, this desired estimate must be a function of two variables a and b. This function describes the properties of "and" and is therefore sometimes called an "and" function. Let us denote this function by $f_\wedge(a, b)$.

What are the reasonable properties of this function?

First property of an "and" operation: monotonicity. First, if our degree of certainty in one or both of the statements S_a and S_b increases, the resulting degree of belief in $S_a \wedge S_b$ should also increase – or maybe remain the same (but it should not decrease). Thus, the function $f_\wedge(a, b)$ must be a (non-strictly) increasing function of both of its variables: if $a \le a'$ and $b \le b'$, then $f_\wedge(a, b) \le f_\wedge(a', b')$.

Second property: our degree of confidence in $S_a \wedge S_b$ cannot be larger than the degree of confidence in S_a. Second, since $S_a \wedge S_b$ implies S_a, our degree of belief in S_a must be larger than (or at least equal to) the degree of belief in $S_a \wedge S_b$. Formally, we must have $f_\wedge(a, b) \le a$. Similarly, the degree of belief in S_b must be larger than (or at least equal to) the degree of belief in $S_a \wedge S_b$, so we must have $f_\wedge(a, b) \le b$.

Third property: $S_a \wedge S_a$. Third, for every statement S_a, "S_a and S_a" means the same as S_a. In terms of the function $f_\wedge$, this means that $f_\wedge(a, a) = a$ for all a.

Resulting "and" operation: derivation and result. Let us show how these three requirements lead to a definition selection of the estimation function $f_\wedge(a, b)$. Let us first consider the case when $a \le b$. In this case, due to monotonicity, we have $f_\wedge(a, a) \le f_\wedge(a, b)$.