8706234

Hankerson, Darrel Richard

BOUNDARY VALUE PROBLEMS FOR N-TH ORDER DIFFERENCE EQUATIONS

The University of Nebraska - Lincoln PH.D. 1986

University
Microfilms
International 300 N. Zeeb Road, Ann Arbor, MI 48106

PLEASE NOTE:

In all cases this material has been filmed in the best possible way from the available copy. Problems encountered with this document have been identified here with a check mark _√_.

1. Glossy photographs or pages _____
2. Colored illustrations, paper or print _____
3. Photographs with dark background _____
4. Illustrations are poor copy _____
5. Pages with black marks, not original copy _____
6. Print shows through as there is text on both sides of page _____
7. Indistinct, broken or small print on several pages _√_
8. Print exceeds margin requirements _____
9. Tightly bound copy with print lost in spine _____
10. Computer printout pages with indistinct print _____
11. Page(s) _____ lacking when material received, and not available from school or author.
12. Page(s) _____ seem to be missing in numbering only as text follows.
13. Two pages numbered _____. Text follows.
14. Curling and wrinkled pages _____
15. Dissertation contains pages with print at a slant, filmed as received _√_
16. Other _____

University
Microfilms
International

BOUNDARY VALUE PROBLEMS FOR n-th ORDER DIFFERENCE EQUATIONS

by

Darrel R. Hankerson

A DISSERTATION

Presented to the Faculty of

The Graduate College in the University of Nebraska

In Partial Fulfillment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics and Statistics

Under the Supervision of Professor Allan C. Peterson

Lincoln, Nebraska

December, 1986

TITLE

Boundary Value Problems for n-th

Order Difference Equations

BY

Darrel R. Hankerson

APPROVED	DATE
Dr. Allan C. Peterson	December 11, 1986
Dr. Gary H. Meisters	December 11, 1986
Dr. Gordon S. Woodward	December 11, 1986
Dr. Thomas S. Shores	December 11, 1986
Dr. Seh-leh Chou	December 11, 1986

SUPERVISORY COMMITTEE

GRADUATE COLLEGE **UNIVERSITY OF NEBRASKA**

GRAD 82002 300 5-66(T)

BOUNDARY VALUE PROBLEMS FOR n-th ORDER DIFFERENCE EQUATIONS

Darrel R. Hankerson, Ph.D.

University of Nebraska, 1986

Adviser: Allan C. Peterson

We are concerned with solutions to the difference equation $Py(t-k) = f(t, y(t))$ where $Py(t-k) = \sum_{i=0}^{n} \alpha_i(t) y(t-k+i)$. Here, k and n are fixed integers with $0 \leq k < n$, and the coefficients $\alpha_i(t)$ are defined on $I+k$ where I is an interval of integers of the form $[a, b]$ or $[a, \infty)$. $Py(t-k) = 0$ is said to be disconjugate on J provided no nontrivial solution has n zeros on J. We will be interested in right and left $(j, n-j)$-disconjugacy as defined by Peterson. We obtain a partial factorization for P if P is right $(j, n-j)$-disconjugate and results relating right and left $(j, n-j)$-disconjugacy are given. An adjoint to $Py(t) = 0$ and disconjugacy properties for the adjoint equation are discussed.

Next, the equation $Ly(t) + p(t)y(t) = 0$ is considered, where L is a disconjugate operator. This work is motivated by results of Elias for the corresponding differential equation. A theorem which bounds the number of certain types of zeros for solutions on an interval is obtained. Using this, sign conditions on $p(t)$ are determined that will guarantee that $Ly(t) + p(t)y(t) = 0$ is right $(k, n-k)$-disconjugate, and a uniqueness theorem for solutions to certain types of boundary value problems is given. Several results give some properties of solutions to this difference equation. Furthermore, a classification of solutions is obtained based on their behavior in a neighborhood of infinity.

Finally, we consider the nonlinear equation $Py(t-k) = f(t, y(t))$. A comparison theorem for solutions of related linear inequalities is obtained. This leads to some disconjugacy results. The Brouwer invariance of domain theorem is used to establish some results on the $(j, n-j)$-problem and its related variational equation. Then we show that under suitable conditions on f and certain related linear equations that the $(n-2, 2)$-

boundary value problem has a unique solution.

Acknowledgements

I wish to express my sincere appreciation to Professor Allan C. Peterson for the advice, encouragement, and the many hours of his time he has given me during the preparation of this paper.

Contents

INTRODUCTION

We are concerned with solutions to the difference equation $Py(t-k) = f(t, y(t))$ where $Py(t-k) = \sum_{i=0}^{n} \alpha_i(t) y(t-k+i)$ and for each fixed t, $f(t,y)$ is a continuous function of y. Here, k and n are fixed integers with $0 \leq k < n$, and the coefficients $\alpha_i(t)$ are defined on $I+k$ where I is an interval of integers of the form $[a,b]$ or $[a,\infty)$, $\alpha_n(t) \equiv 1$ and $(-1)^n \alpha_0(t) > 0$. Solutions to this difference equation are defined on I^n where $I^n = [a, b+n]$ if $I = [a,b]$, and $I^n = I$ if $I = [a,\infty)$. In Chapters 1–3, we study linear difference equations.

Chapter 1 includes preliminary results on the equation $Py(t-k) = h(t)$. Two variation of constant formulas due to Peterson are given, and characterizations of the Green's function for the boundary value problem

$$Py(t-k) = h(t), \quad t \in I+k$$
$$\Delta^i y(a) = 0, \quad 0 \leq i \leq j-1$$
$$\Delta^i y(b+j+1) = 0, \quad 0 \leq i \leq n-j-1$$

are given. We also define an adjoint equation to $Py(t) = 0$ and relationships between solutions of the equation and its adjoint are discussed. The adjoint theory proved useful in Peterson [33].

As defined by Hartman [19], $Py(t-k) = 0$ is said to be disconjugate on an interval J provided no nontrivial solution has n zeros on J. A simple example of a disconjugate equation is given by $\Delta^n y(t) = 0$. Hartman has shown that $Py(t-k) = 0$ is disconjugate on J if and only if P has a certain factorization. Further, necessary and sufficient conditions for disconjugacy in terms of the coefficients $\alpha_i(t)$ and sign conditions on the Green's functions in the case that P is disconjugate are given by Hartman. In Chapter 2,

we will be interested in right and left $(j, n-j)$-disconjugacy as defined by Peterson. We list these definitions here.

DEFINITION. Let J be a subinterval of I^n with card $J \geq n$, and let $1 \leq j \leq n-1$. We say that $Py(t-k) = 0$ is ***right*** $(j, n-j)$***-disconjugate*** on J provided there is no nontrivial solution $y(t)$ and integers α, $\beta \in J$ with $\alpha < \alpha + j \leq \beta \leq \beta + n - j - 1 \in J$ such that

$$y(\alpha + i) = 0, \quad 0 \leq i \leq j-1$$

$$y(\beta + i) = 0, \quad 0 \leq i \leq n-j-2 \quad (\text{if } n-j \geq 2)$$

and y has a generalized zero at $\beta + n - j - 1$. Similarly, we say that $Py(t-k) = 0$ is *left* $(j, n-j)$*-disconjugate* on J provided there is no nontrivial solution $y(t)$ and integers α, $\beta \in J$ with $\alpha < \alpha + j \leq \beta \leq \beta + n - j - 1 \in J$ such that

$$y(\alpha + i) = 0, \quad 0 \leq i \leq j-2 \quad (\text{if } j \geq 2)$$

$$y(\beta + i) = 0, \quad 0 \leq i \leq n-j-1$$

and y has a generalized zero at $\alpha + j - 1$.

Peterson has obtained necessary conditions for $(j, n-j)$-disconjugacy of $Py(t-k) = 0$ in terms of the coefficients of P. We obtain a partial factorization for P if P is right $(j, n-j)$-disconjugate and we show that $Py(t-k) = 0$ is disconjugate iff $Py(t-k) = 0$ is right $(j, n-j)$-disconjugate for $1 \leq j \leq n-1$. Results relating right and left $(j, n-j)$-disconjugacy are given. Right and left $(j, n-j)$-disconjugacy for the adjoint equation are defined, and disconjugacy properties for the adjoint equation are discussed.

Chapter 3 was done jointly with Allan Peterson. The equation $Ly(t) + p(t)y(t) = 0$, $t \in [a, \infty)$, is considered where L is a disconjugate operator. This work is motivated by results of Elias for the corresponding differential equation. One of the first results is a somewhat technical theorem which bounds the number of certain types of zeros

for solutions on an interval $[a, b]$. Using this, sign conditions on $p(t)$ are determined that will guarantee that $Ly(t) + p(t)y(t) = 0$ is right $(k, n-k)$-disconjugate, and a uniqueness theorem for solutions to certain types of boundary value problems is given. Several other results give properties of solutions to this difference equation. Furthermore, a classification of solutions is obtained based on their behavior in a neighborhood of infinity.

Finally, in Chapter 4 we consider the nonlinear equation $Py(t-k) = f(t, y(t))$. We assume that for each fixed t, $f(t, y)$ is a continuous function of y. In Peterson [34], a comparison theorem for solutions of $Pu(t-k) \geq q_1(t)u(t)$ and $Pv(t-k) \leq q_2(t)v(t)$ is given for the case $q_1(t) \geq q_2(t)$ and $Pv(t-k) = q_2(t)v(t)$ is right $(n-1, 1)$-disconjugate. Using this, an existence-uniqueness theorem is obtained for the $(n-1, 1)$-boundary value problem

$$Py(t-k) = f(t, y(t))$$
$$\Delta^i y(a) = A_i, \quad 0 \leq i \leq n-2$$
$$y(b+n) = B$$

provided there is a function $q(t)$ such that $f(t, u) - f(t, v) \geq q(t)[u - v]$ and the difference equation $Py(t-k) = q(t)y(t)$ is right $(n-1, 1)$-disconjugate.

We obtain similar results for the $(n-2, 2)$-problem under suitable disconjugacy assumptions on the equations $Pu(t-k) = q_1(t)u(t)$ and $Pv(t-k) = q_2(t)v(t)$. First, comparison results are obtained for solutions to the inequalities. Some results for $(j, n-j)$-boundary value problems and their related variational equations are established using the Brouwer invariance of domain theorem. Then under suitable conditions on f, the existence-uniqueness theorem is established for the $(n-2, 2)$-problem.

Chapter 1

Preliminary Results

In this chapter we will define the n-th order linear difference equation and its adjoint, list some definitions used throughout the paper, and provide some results on the difference equation and its adjoint. Much of the notation is from Hartman [**19**] and Peterson (for example, [**31**], [**32**]). In general, the interval notation in this paper specifies an interval of integers, $n \geq 2$ is an integer, and in this chapter, k is a fixed integer with $0 \leq k \leq n-1$. If $I = [a,b]$, let I^j be the interval $[a, b+j]$ and let $I+j = [a+j, b+j]$. If $I = [a,\infty)$, let $I^j = I$ and let $I+j = [a+j,\infty)$.

For the nth order linear difference equation

$$Py(t-k) \equiv \sum_{i=0}^{n} \alpha_i(t) y(t-k+i) = 0, \quad t \in I+k \tag{1.1}$$

we assume the coefficients $\alpha_i(t)$ are defined on $I+k$, $\alpha_n(t) \equiv 1$, and $\alpha_0(t)$ satisfies

$$(-1)^n \alpha_0(t) > 0 \tag{1.2}$$

for $t \in I+k$. Solutions to the difference equation are defined on I^n.

Condition (1.2) implies $\alpha_0(t) \neq 0$ which guarantees that solutions to the initial value problem

$$\begin{aligned} Py(t-k) &= h(t) \\ y(t_0+i) &= y_i, \quad 0 \leq i \leq n-1 \end{aligned}$$

for $t_0 \in I$ exist on I^n, and that (1.1) has exactly n linearly independent solutions on I^n. We will see (as noted by Hartman [**19**]) that (1.2) is a necessary condition for (1.1)

to be disconjugate and that (1.2) insures that the Wronskian of n linearly independent solutions is of one sign.

Define the difference operator Δ by $\Delta y(t) = y(t+1) - y(t)$. Then recursively define the operators Δ^i by $\Delta^i y(t) = \Delta(\Delta^{i-1} y(t))$ for $i = 1, 2, \ldots$, where $\Delta^0 y(t) = y(t)$.

DEFINITION 1.1. (Hartman [**19**]) Let $y(t)$ be a solution of (1.1). Then y has a *node* at t_0 if $y(t_0) = 0$ or, if $t_0 > a$, $y(t_0 - 1)y(t_0) < 0$. We say y has a *generalized zero* at t_0 in case either $y(t_0) = 0$ or there exists an integer j with $1 \leq j \leq t_0 - a$ such that

$$(-1)^j y(t_0 - j)y(t_0) > 0$$

$$y(t) = 0, \quad t_0 - j < t < t_0 \text{ (if } j > 1).$$

Note that every node is a generalized zero. The difference equation (1.1) is called *r-disconjugate* if no solution $y \not\equiv 0$ has n nodes on I^n. We say that (1.1) is *disconjugate* on I^n if no solution $y \not\equiv 0$ has n generalized zeros on I^n.

To see that (1.2) is a necessary condition for (1.1) to be disconjugate on I^n, let $t_0 \in I+k$ and let $y(t)$ be the solution of (1.1) satisfying

$$y(t_0 - k) = 1$$

$$y(t_0 - k + i) = 0, \quad 1 \leq i \leq n - 1.$$

Then $y(t_0 - k + n) = -\alpha_0(t_0)y(t_0 - k) = -\alpha_0(t_0)$ so that

$$(-1)^n y(t_0 - k)y(t_0 - k + n) = -(-1)^n \alpha_0(t_0).$$

If $(-1)^n \alpha_0(t_0) \leq 0$ then y has a generalized zero at $t_0 - k + n$. In this case, y has n generalized zeros on I^n and (1.1) is not disconjugate on I^n.

Hartman [**19**, Theorem 5.1] has shown that disconjugacy and r-disconjugacy are equivalent. Further, (1.1) is disconjugate on I^n if and only if there exist positive functions $p_j(t)$, $t \in I^{n-j}$, for $0 \leq j \leq n$ such that we have the "Polya" factorization (see [**35**])

$$Pu = p_n \Delta\{p_{n-1}\Delta[\cdots\Delta(p_0 u)]\}.$$

For functions $y_1, \ldots, y_j$ defined on $\{t_1, \ldots, t_j\}$, define

$$W(y_1, \ldots, y_j)(t_1, \ldots, t_j) = \begin{vmatrix} y_1(t_1) & \cdots & y_j(t_1) \\ y_1(t_2) & \cdots & y_j(t_2) \\ \vdots & \ddots & \vdots \\ y_1(t_j) & \cdots & y_j(t_j) \end{vmatrix}.$$

If $t_1 = t$, $t_2 = t+1, \ldots, t_j = t+j-1$, we write

$$W(y_1, \ldots, y_j)(t) \equiv W(y_1(t), \ldots, y_j(t)) \equiv W(y_1, \ldots, y_j)(t_1, \ldots, t_j)$$

and we call this determinant the *Wronskian* of $y_1, \ldots, y_j$ at t. It is easy to show that

$$W(y_1, \ldots, y_j)(t) = \begin{vmatrix} y_1(t) & \cdots & y_j(t) \\ \Delta y_1(t) & \cdots & \Delta y_j(t) \\ \vdots & \ddots & \vdots \\ \Delta^{j-1} y_1(t) & \cdots & \Delta^{j-1} y_j(t) \end{vmatrix}.$$

Let J be a subinterval of I with card $J \geq n-1$. We say that the set of functions $y_1, \ldots, y_{n-1}$ is a *$w_n(J)$-system on J* (see [**19**]) if they are defined on J and

$$w_j(t) \equiv W(y_1, \ldots, y_j)(t) > 0$$

for $t \in J^{1-j}$, $1 \leq j \leq n-1$.

A well known result (for example, see [**19**, Proposition 2.7]) is given in the next proposition.

PROPOSITION 1.1. *(Liouville's formula) Let $y_1(t), \ldots, y_n(t)$ be solutions of* (1.1). *Then*

$$W(y_1, \ldots, y_n)(t+1) = (-1)^n \alpha_0(t+k) W(y_1, \ldots, y_n)(t), \quad t \in I.$$

In particular, if $y_1, \ldots, y_n$ are linearly independent solutions of (1.1) then Proposition 1.1 shows that the Wronskian of n linearly independent solutions of (1.1) is of one sign on I since (1.2) holds. This is another reason why we assume that (1.2) holds.

We will use two variation of constants formulas given in [**34**]. For the sake of completeness, we include a proof of the first case of this result. In any of the sums $\sum_{\alpha}^{\beta}$, if $\beta < \alpha$ we define the sum to be zero.

PROPOSITION 1.2. *Assume $I = [a, b]$.*

(i) *For each fixed $s \in [a, b+1]$ let $U(t, s)$ be the solution of (1.1) satisfying*

$$U(s+i, s) = 0, \quad 0 \le i \le n-2$$

$$U(s+n-1, s) = 1.$$

The solution of the initial value problem

$$Py(t-k) = h(t)$$

$$y(a+i) = 0, \quad 0 \le i \le n-1$$

is given by

$$y(t) = \sum_{s=a+k}^{t+k-1} U(t, s-k+1)h(s) = \sum_{s=a+k}^{t+k-n} U(t, s-k+1)h(s).$$

(If $s > b+k$ in this sum then the corresponding terms are zero.)

(ii) *For each fixed $s \in [a+n-1, b+n-1]$ let $V(t, s)$ be the solution of (1.1) satisfying*

$$V(s-i, s) = 0, \quad 0 \le i \le n-2$$

$$V(s-n+1, s) = 1.$$

The solution of the initial value problem

$$Py(t-k) = h(t)$$

$$y(b+n-i) = 0, \quad 0 \le i \le n-1$$

is given by

$$y(t) = \sum_{s=t-n+k+1}^{b+k} V(t, s+n-k-1)\frac{h(s)}{\alpha_0(s)}.$$

(If $s < a+k$ in this sum then the corresponding terms are zero.)

PROOF: (i) Set

$$y(t) = \sum_{s=a+k}^{t+k-1} U(t, s-k+1)h(s), \quad a \le t \le b+n.$$

Then

$$y(t-k)=\sum_{s=a+k}^{t-1}U(t-k,s-k+1)h(s)$$

$$y(t-k+1)=\sum_{s=a+k}^{t-1}U(t-k+1,s-k+1)h(s)+\underbrace{U(t-k+1,t-k+1)}_{=0}h(t)$$

$$\cdots\quad\cdots$$

$$\begin{aligned}y(t-k+n)&=\sum_{s=a+k}^{t-1}U(t-k+n,s-k+1)h(s)+U(t-k+n,t-k+1)h(t)\\&\quad+\underbrace{U(t-k+n,t-k+2)}_{=0}h(t+1)\\&\quad+\cdots+\underbrace{U(t-k+n,t-k+n-1)}_{=0}h(t-n+1)\\&=\sum_{s=a+k}^{t-1}U(t-k+n,s-k+1)h(s)+h(t).\end{aligned}$$

Multiply $y(t-k+i)$ by $\alpha_i(t)$, $0\le i\le n$, to get

$$Py(t-k)=\sum_{s=a+k}^{t-1}PU(t-k,s-k+1)h(s)+h(t).$$

$U(t,s)$ is a solution of $Py(t-k)=0$ so the above is

$$Py(t-k)=h(t)$$

and letting $t=a+k$ in the equations at the beginning of the proof shows

$$y(a+i)=0,\quad 0\le i\le n-1.$$

We will also be interested in solving certain boundary value problems of the form

$$\begin{aligned}Py(t-k)&=h(t),\quad a+k\le t\le b+k\\y(a+i)&=0,\qquad 0\le i\le j-1\\y(b+n-i)&=0,\qquad 0\le i\le n-j-1\end{aligned}\tag{1.3}$$