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by

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AUTOMORPHISM GROUPS OF SEMIGROUPS

BY

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Paul L. Dussere

Contents

Chapter I

Introduction

This dissertation is concerned with two general problems:

(I) Given a class $K = \{G_\alpha\}_{\alpha \varepsilon A}$ of groups with some common property, does there exist a class $K' = \{S_\beta\}_{\beta \varepsilon B}$ of semigroups such that for each G_α in K there is an S_β in K' whose automorphism group is isomorphic to G_α?

(II) Given a class $L = \{S_\alpha\}_{\alpha \varepsilon A}$ of semigroups with some common property, what can be said about the class $L' = \{G_\alpha\}_{\alpha \varepsilon A'}$ where for each α in A, G_α is the automorphism group of S_α?

It is well known that not every group can be realized as the automorphism group of some group. For example, the finite cyclic groups of odd order (greater than 1), the infinite cyclic group, and the group of rational numbers under addition are not the automorphism group of any group. (The proof of this assertion will appear at the end of the introduction.) On the other hand, de Groot [3](1) has shown that every group is the

(1)The square brackets refer to entries in the bibliography.

automorphism group of a commutative ring with unit. His methods are topological, in that he represents a given group as the group of homeomorphisms of a suitable topological space onto itself. The algebraic corollary arises from using the ring of continuous functions defined on the space. He also shows that every group is the automorphism group of some graph. Birkhoff [1] has shown that every group is the automorphism group of some distributive lattice. However de Groot's results show that not all groups are included among the automorphism groups of Boolean algebras, i.e. complemented distributive lattices.

Some special cases of Problem I are considered in Chapters II and III. If K is the class of all groups of order less than or equal to c, the cardinality of the real numbers, then a corresponding class K' is constructed in Theorem 2.4. The order of each S_α in K' is less than or equal to 2^c. In Theorem 2.6 K consists of all finite groups, and a class K' consisting of finite semigroups is constructed. In Chapter III the effect of restricting K to certain classes of abelian groups is studied. It is shown in Corollary 3.3 and Corollary 3.7 that every finite abelian group can be realized as the automorphism group of a finite commutative semigroup and every finitely generated abelian group can be realized as the automorphism group of a countable commutative semigroup. The remainder of

Chapter III, culminating in Theorem 3.10, is concerned with the representation of decomposable groups as automorphism groups of semigroups, assuming the direct factors can themselves be realized as automorphism groups of semigroups. The result is effected for finite direct products. The construction is such that if the factors can be realized as automorphism groups of commutative semigroups then the direct product of the factors is the automorphism group of a commutative semigroup.

Two cases of Problem II are studied in Chapters IV and V. In Chapter IV the case in which **L** is the class of semigroups which are the multiplicative semigroup J_m of the integers modulo m is considered. It is shown that it is sufficient to consider the case $m = p^n$ where p is a prime and n is a positive integer. If p is odd a description of the automorphism group of J_m ($m = p^n$) is given. In Chapter V, **L** is the class of completely 0-simple semigroups having sandwich matrix all of whose non-zero entries equal the identity of the structure group. A description of L' is given. This is then used to determine the automorphism groups of Brandt semigroups, rectangular bands, and right groups.

The following notation will be used throughout the dissertation:

All semigroups will be written multiplicatively

unless otherwise stated. The automorphism group of a semigroup S will be denoted by Aut(S). The symbol I (or I_W, where W is the domain of I) will be used for the identity mapping. All mappings will be single valued. The cardinal number of the set A will be denoted by $|A|$. J and J^+ will denote the integers and the positive integers, respectively. If α is a mapping whose domain includes A then $\alpha|A$ will mean the restriction of α to A. If a is an element of the group G then $\langle a \rangle$ will denote the cyclic subgroup of G generated by a. If A and B are sets then $A \setminus B = \{x \in A | x \notin B\}$. If $\{S_i\}_{i \in M}$ is a collection of semigroups then $\prod_{i \in M} S_i$ denotes the direct product of the S_i. If $s_1, s_2, \ldots, s_r$ are elements of a semigroup S then $\prod_{i=1}^{r} s_i$ denotes the product $s_1 s_2 \ldots s_r$ in S. If $c_1, c_2, \ldots, c_r$ is a finite collection of cardinal numbers then $\prod_{i=1}^{r} c_i$ denotes the product of the cardinal numbers. The symbol $||$ will be used at the end of a proof.

Now it will be shown that:

i) Finite cyclic groups of odd order (greater than 1) and the infinite cyclic group are not automorphism groups of any group.

ii) The additive group of rationals is not the automorphism group of any group.

Proof of i) (The proof follows Kurosh [4].) Let G be a group such that Aut(G) $\cong$ C where C is a non-trivial cyclic group. Then the inner automorphism group of G is cyclic so G/Z is cyclic. Let $x \varepsilon G$ be in a coset xZ that generates G/Z. Then the subgroup of G generated by x and Z is all of G. But this subgroup is commutative so G = Z. Now since G is abelian and $|G| > 2$, G has an automorphism of order 2. Hence Aut(G) cannot be of finite odd order and it cannot be the infinite cyclic group. ||

Proof of ii) Let <R;+> denote the group of rational numbers under addition. Suppose G is a multiplicative group such that Aut(G) $\cong$ <R;+>. Then since G/Z is isomorphic to the inner automorphism group of G, it is isomorphic to a subgroup of <R;+>. But it is well known [7, p. 52, exercise 3.2.20] that this is possible only for the trivial subgroup of <R;+>. Hence G is abelian. But then, as in i), G has an automorphism of order 2. However <R;+> contains no element of order 2 so there is no group G such that Aut(G) $\cong$ <R;+>. ||

Chapter II

What Groups Occur as Automorphism Groups of Semigroups?

Definition. Let G be a group and T a subset of G. Then the normalizer of T in G, denoted by $N_G(T)$ is defined by

$$N_G(T) = \{x \in G \mid x^{-1}Tx = T\}.$$

When no ambiguity arises from abbreviation, $N_G(T)$ will be abbreviated $N(T)$.

The centralizer of T in G, denoted by $C_G(T)$ is defined by

$$C_G(T) = \{x \in G \mid x^{-1}tx = t \text{ for every } t \in T\}.$$

Again, $C(T)$ will be used as an abbreviation of $C_G(T)$ when no ambiguity arises from this.

The centralizer of G in G, $C_G(G)$ is called the center of G and will be denoted by Z.

Definition. A group G is said to be complete if the center of G is trivial and every automorphism of G is inner.

Let U be a semigroup and t be an integer greater than 1. Let B be an index set and let

$R = \{r_\beta^{(1)}\} \cup \{r_\beta^{(2)}\} \cup \ldots \cup \{r_\beta^{(t-1)}\}$ (where β ranges over B)

be a collection of $(t-1)\cdot|B|$ distinct symbols such that

$R \cap U = \emptyset$. Let Θ be a mapping of B into U and let $S = R \cup U$. Define a binary operation $(\circ)$ on S by

$$\begin{cases} u \circ v = u \cdot v \quad \text{for every } u, v \in U \\ r_\beta^{(i)} \circ u = (\beta\Theta)^i \cdot u \quad \text{for every } u \in U,\ \beta \in B,\ i = 1,2,\ldots,t-1 \\ u \circ r_\beta^{(i)} = u \cdot (\beta\Theta)^i \quad \text{for every } u \in U,\ \beta \in B,\ i = 1,2,\ldots,t-1 \\ r_\beta^{(i)} \circ r_\delta^{(j)} = (\beta\Theta)^i \cdot (\delta\Theta)^j \quad \text{for every } \beta, \delta \in B,\ \beta \neq \delta \\ \qquad\qquad\qquad\qquad\qquad\qquad 1 \leq i, j < t \\ (r_\beta^{(i)})^j = \begin{cases} r_\beta^{(ij)} & \text{if } ij < t \\ (\beta\Theta)^{ij} & \text{if } ij \geq t \end{cases} \quad \begin{matrix} i = 1,2,\ldots,t-1 \\ j = 1,2,\ldots \end{matrix} \end{cases}$$

where positive powers of an arbitrary $s \in S$ are defined recursively by $s^2 = s \circ s$, $s^{n+1} = s^n \circ s$, $n = 2,3,\ldots$.

By calculation of the products involved it is easily seen that $\langle S;\circ\rangle$ is a semigroup. In the sequel the operation $(\circ)$ will be replaced by juxtaposition of symbols.

Definition. The element $r_\beta^{(1)}$ is said to be a replica of the element $\beta\Theta$ of U of degree t.

Lemma 2.1. Let S_M denote the group of all permutations of a fixed set of cardinality M, $M \neq 1,2,6$. Let G be a subgroup of S_M and λ an element of S_M such that

i) $C_{S_M}(\lambda) \cap G = \{I\}$;

ii) there exists an element σ_λ of S_M for which $\sigma_\lambda^2 = \lambda$;

iii) $G = N_{S_M}(T)$, where $T = \{\gamma^{-1}\lambda\gamma \mid \gamma \in G\}$.

Then there exists a semigroup S containing S_M as a subsemigroup such that Aut(S) is isomorphic to G.

Proof: Let $\rho, \mu \in G$ be such that $\rho^{-1}\lambda\rho = \mu^{-1}\lambda\mu$. Then $\mu\rho^{-1}\lambda = \lambda\mu\rho^{-1}$ so $\rho\mu^{-1} \in C(\lambda)$, the centralizer of λ in S_M. But $\rho\mu^{-1} \in G$ so by i), $\rho\mu^{-1} = I$ and $\rho = \mu$. Hence to each $\omega \in T$ there corresponds a unique $\delta \in G$ such that $\omega = \delta^{-1}\lambda\delta$. For each $\omega = \delta^{-1}\lambda\delta \in T$ define $\sigma_\omega = \delta^{-1}\sigma_\lambda\delta$. Then $\sigma_\omega \in S_M$ and $\sigma_\omega^2 = \delta^{-1}\sigma_\lambda^2\delta = \delta^{-1}\lambda\delta = \omega$.

Let $R = \{\rho_\omega \mid \omega \in T\}$ be a collection of $|T|$ distinct symbols such that $R \cap S_M = \emptyset$. Denote $R \cup S_M$ by S and extend the definition of multiplication from S_M to S by regarding each ρ_ω as a replica of σ_ω of degree 2, i.e., the mapping Θ in the definition of replica is $\Theta\colon \omega \to \sigma_\omega$ for every $\omega \in T$. Thus ρ_ω multiplies as if it were σ_ω. Then S under this (multiplicative) operation is a semigroup.

Let $\alpha \in \mathrm{Aut}(S)$. Then $\alpha \mid S_M \in \mathrm{Aut}(S_M)$ since S_M is the unique maximal subgroup of S. Since $M \neq 1, 2, 6$, S_M is complete [7, p. 452]. Therefore there is an x in S_M such that $\alpha|S_M$ is the inner automorphism determined by x, i.e.

$\alpha|S_M\colon a \to x^{-1}ax$ for every $a \in S_M$.

Since α permutes S_M it also permutes R. Hence it permutes $\{\rho_\omega^2 | \omega \in T\} = T$. Thus for every $\omega \in T$, $x^{-1}\omega x = \omega(\alpha|S_M) = \omega\alpha \in T$, i.e. $x \in N(T)$, the normalizer of

T in S_M.

Let $\rho_\omega \in R$, $\rho_\omega\alpha = \rho_\pi$. Then $\omega\alpha = \sigma_\omega^2\alpha = \rho_\omega^2\alpha = (\rho_\omega\alpha)^2 = \rho_\pi^2 = \sigma_\pi^2 = \pi$. But $\omega \in S_M$ so $\pi = \omega\alpha = x^{-1}\omega x$.

Thus every $\alpha \in \mathrm{Aut}(S)$ is of the following form for some $x \in G$:

$$(2.1) \qquad \alpha = \alpha_x: \begin{cases} y \to x^{-1}yx & \text{for every } y \in S_M \\ \rho_\omega \to \rho_{x^{-1}\omega x} & \text{for every } \rho_\omega \in R \end{cases}$$

Conversely let $x \in N(T) = G$ and define α_x by (2.1). Clearly α_x is a mapping of S onto S. It will be shown that $\alpha_x \in \mathrm{Aut}(S)$. First the homomorphism property will be checked.

Let $x \in G$ and $\omega \in T$. Then there is a unique element $\delta \in G$ such that $\omega = \delta^{-1}\lambda\delta$. Now using the definition of σ_ω and the fact that $\sigma_\omega \in S_M$:

$$\sigma_\omega\alpha_x = x^{-1}\sigma_\omega x = x^{-1}\delta^{-1}\sigma_\lambda\delta x = (\delta x)^{-1}\sigma_\lambda(\delta x)$$

$$= \sigma_{(\delta x)^{-1}\lambda(\delta x)} = \sigma_{x^{-1}(\delta^{-1}\lambda\delta)x} = \sigma_{x^{-1}\omega x} \quad \text{so}$$

$$(2.2) \qquad \sigma_\omega\alpha_x = \sigma_{x^{-1}\omega x} \quad \text{for every } x \in G,\ \omega \in T.$$

Now let $a,b \in S$.

Case i) $a,b \in S_M$. Since $\alpha_x|S_M$ is an inner automorphism, $(ab)\alpha_x = (a\alpha_x)(b\alpha_x)$.

Case ii) $a \in S_M$, $b = \rho_\omega \in R$. Then using case i) and (2.2), $(ab)\alpha_x = (a\rho_\omega)\alpha_x = (a\sigma_\omega)\alpha_x$

$= (a\alpha_x)(\sigma_\omega\alpha_x) = (a\alpha_x)(\sigma_{x^{-1}\omega x})$

$= (a\alpha_x)(\rho_{x^{-1}\omega x}) = (a\alpha_x)(\rho_\omega\alpha_x)$

$= (a\alpha_x)(b\alpha_x)$. Similarly $(ba)\alpha_x$

$= (b\alpha_x)(a\alpha_x)$.

Case iii) $a = \rho_\omega \in R$, $b = \rho_\pi \in R$. Then using case i) and (2.2), $(ab)\alpha_x = (\rho_\omega\rho_\pi)\alpha_x = (\sigma_\omega\sigma_\pi)\alpha_x$

$= (\sigma_\omega\alpha_x)(\sigma_\pi\alpha_x) = \sigma_{x^{-1}\omega x}\sigma_{x^{-1}\pi x} = \rho_{x^{-1}\omega x}\rho_{x^{-1}\pi x}$

$= (\rho_\omega\alpha_x)(\rho_\pi\alpha_x) = (a\alpha_x)(b\alpha_x)$. Hence α_x is a homomorphism of S.

Since $\alpha_x|S_M$ is an inner automorphism, it is 1-1 on S_M. Suppose there are $\rho_\omega, \rho_\pi \in R$ such that $\rho_\omega\alpha_x = \rho_\pi\alpha_x$. Then $\rho_{x^{-1}\omega x} = \rho_{x^{-1}\pi x}$ so $x^{-1}\omega x = x^{-1}\pi x$. Hence $\omega = \pi$ and $\rho_\omega = \rho_\pi$ so α_x is 1-1 on all of S. Hence α_x is an automorphism of S.

Now define the mapping Θ by

(2.3) $\Theta: x \to \alpha_x$ for every $x \in G$

where α_x is defined by (2.1). It has been shown that $\alpha_x \in \mathrm{Aut}(S)$ and that every automorphism of S is of the form (2.1) so Θ is a mapping of G onto Aut(S).

Suppose $x, y \in G$ and $x\Theta = y\Theta$, i.e. $\alpha_x = \alpha_y$. Then for every $a \in S_M$, $a\alpha_x = a\alpha_y$ so $x^{-1}ax = y^{-1}ay$ and hence $xy^{-1}a = axy^{-1}$. Thus for every $a \in S_M$, $xy^{-1} \in C(a)$ so