HOMOLOGICAL CHARACTERIZATIONS OF QUASI-COMPLETE INTERSECTIONS

by

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A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Luchezar L. Avramov and Srikanth B. Iyengar

Lincoln, Nebraska

August, 2016

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University of Nebraska, 2016

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Let R be a commutative ring, $(\boldsymbol{f})$ an ideal of R, and $E = K(\boldsymbol{f}; R)$ the Koszul complex. We investigate the structure of the Tate construction T associated with E. In particular, we study the relationship between the homology of T, the quasi-complete intersection property of ideals, and the complete intersection property of (local) rings.

ACKNOWLEDGMENTS

First of all, I thank my advisors Luchezar Avramov and Srikanth Iyengar. I am very grateful for their support, patience, and guidance. They consistently encouraged and enabled me to be a better researcher and writer.

I thank Brian Harbourne and Mark Walker for their careful readings of this dissertation and their comments on it. I also thank Andy Kustin and Liana Şega for helpful conversations regarding this work.

I have greatly enjoyed my time at the University of Nebraska–Lincoln. I thank the current and past faculty and graduate students who have made my experience a positive one.

I thank Thomas Sibley for introducing me to mathematical research as an undergraduate and Bret Benesh for his support as my undergraduate thesis advisor and for many conversations about teaching. I also thank Gary Brown and Jennifer Galovich encouraging me in my decision to attend graduate school.

DEDICATION

To Kim

GRANT INFORMATION

This research was partially supported through U.S. Department of Education grant P00A120068 (GAANN) and through National Science Foundation Awards DMS-1103176 and DMS-1503044.

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Chapter 1

Introduction

Let R be a commutative ring and I an ideal of R. For a generating set $\boldsymbol{f}$ of I let E denote the Koszul complex $K(\boldsymbol{f}; R)$; its homology $H_*(E)$ is naturally an algebra over the quotient ring R/I. The ideal I is said to be a *quasi-complete intersection* if $H_1(E)$ is free over R/I and $H_*(E)$ has the structure of an exterior algebra on $H_1(E)$; see Definition 3.2.

In the case where $(R, \mathfrak{m})$ is a local (Noetherian) ring and E is the Koszul complex on a minimal generating set of $\mathfrak{m}$, such an algebra structure on $H_*(E)$ appears in [1]. It is shown that this algebra structure is equivalent to the *complete intersection* property of R, and is further related to the homological properties of the *Tate construction*; see Chapter 4. The Tate construction is the second step in a *Tate resolution* of S over R, i.e, it is the result of adjoining (to the Koszul complex E) variables of degree two to annihilate the degree one homology of E; see [25, §2].

As Avramov, Henriques, and Şega [5] note, these ideals were first introduced in Rodicio's paper [22] and in his joint work with Blanco and Majadas [9] as ideals having *free exterior Koszul homology.* The quasi-complete intersection nomenclature is due to Avramov et al. [5, 1.1].

Let $\boldsymbol{z}$ be a set of degree one cycles whose homology classes generate $H_1(E)$ and let T be the Tate construction on $\boldsymbol{f}$ and $\boldsymbol{z}$ (see Construction 2.10). Blanco, Majadas, and Rodicio characterize quasi-complete intersection ideals as follows:

Theorem 1.1 ([10, Theorem 1])**.** *The following conditions are equivalent:*

(1) I is a quasi-complete intersection and $\boldsymbol{z}$ represents a basis for the free S-module $H_1(E)$.

(2) $H_i(T) = 0$ for all $i > 0$.

This dissertation builds on the work in [1], [10], and [17] and makes a contribution to the study of quasi-complete intersection ideals, with applications to the study of (local) complete intersection rings. In particular, we establish results in the following two themes:

(I) *The quasi-complete intersection property of I can be detected from a finite band of vanishing of $H_*(T)$.*

The size and location of the band of vanishing depend on computable numerical invariants of the ideal I. Moreover, more flexibility in both components is possible given mild assumptions on I; see Proposition 3.20 and Theorem 4.9.

(II) *The quasi-complete intersection property of the maximal ideal $\mathfrak{m}$ of a local ring can be detected from the vanishing in a single degree of $H_i(T)$.*

The case $i = 2$ is a result of Assmus [1, Theorem 2.7]; the case $i = 2$ or 3 is addressed in Theorem 4.11. We study the case $i \geq 5$ in the context of rings which are *Golod away from a complete intersection*; see Section 5.2.

This project connects to two earlier bodies of work in which the (eventual) vanishing of the homology of a complex is determined by the vanishing of the complex in a

single degree, namely the theory of Koszul rigidity and the vanishing of the deviations of a local ring. These topics and their connections to this project will be outlined in Section 2.5 and Section 2.6, respectively.

Avramov, Henriques, and Şega [5] present one direction of an ideal-theoretic characterization of quasi-complete intersection ideals, namely that an exact ideal (i.e., an ideal generated by a sequence of *exact elements*) is a quasi-complete intersection; see Section 3.1. The converse does not hold: In [20, Example 4.1] Kustin, Şega, and Vraciu give an example of a quasi-complete intersection which cannot be generated by a sequence of exact elements. In addition, [20, Lemma 1.7] shows that for two-generated ideals, a finite band of vanishing of the homology of the Tate construction is related to the quasi-complete intersection property.; see Section 3.2.1.

Notation. Throughout this work, the following assumptions and notations are in force. All rings are assumed to be commutative and unitary. Our ubiqutous commutative ring is denoted R, I is an ideal of R, and $\boldsymbol{f}$ denotes a generating set of I.