METHOD OF MOMENTS SOLUTIONS

FOR THE ELECTROMAGNETIC FIELDS OF

A RECTANGULAR APERTURE ON A CYLINDRICAL BODY

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To someone I will love in the future

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by

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THESIS

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ABSTRACT

The reaction matching of modes is used to determine the self- admittance of a rectangular slot antenna on a reentry missile in the free-space. Matrix elements for the aperture electric fields are obtained by using the method of moments. The plasma sheath in front of the antenna is modeled by a coupled transmission line. Self-admittance and matrix elements are also derived for the plasma case. Theoretical results and numerical calculations of corresponding physical quantities are compared when the plasma is reduced to free-space.

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Chapter 1

INTRODUCTION

The high speed of a reentry missile into the earth's atmosphere causes tremendous aerodynamic friction between the metal surface of the vehicle and the particles around it. The intense heating from this friction ionizes the particles and results in a plasma sheath. Transmission of signals to and from the missile is affected by the interaction of electromagnetic waves and the plasma particles. One important interference considered in this thesis is the reflection of the energy coming from a rectangular slot antenna on the circular cylindrical body of the missile. This reflection weakens the transmission of signals. The plasma thus acts as a dismatched load for the antenna. The reflection is determined by the input impedance or admittance seen by the antenna at the aperture. Knowing either of these quantities will allow one to design an impedance-matched antenna so that maximum transfer of power can be obtained. In this thesis only the incident and reflected dominant mode are considered. The self-admittance and reflection coefficient are calculated for the plasma with different dielectrics and for different radii and plasma thickness. These results are compared with those obtained previously by others [1,2]. Different formulations

of similar problems have been approached by several authors [3,4,5,6,7]. In this thesis a coupled transmission line proposed by Stewart and Golden [8] is used. However the resultant coupling coefficients are modified to avoid the loss of significance. The inhomogeneous plasma layer that covers the aperture is broken up into a number of thin homogeneous layers, each of which is represented by a section of coupled TE amd TM transmission line characterized by voltages and currents. These voltages and currents in each section are related to those in next section in a coupled way. This relation is mathematically transformed into a matrix. Thus the voltages and currents at the aperture are related to those outside the plasma layer or in the free space by a product of matrices, each of which corresponds to a homogeneous layer. Once the voltages and currents are known at the aperture, the self-admittance and reflection coefficient can be determined.

In Chapter 2, basic equations of cylindrical modal analysis is presented. Using these equations, field quantities external to the slot antenna are expressed in terms of known aperture field components. In Chapter 3, matrix elements for aperture tangential electric field components are derived and free-space self-admittances for axial and azimuthal orientations are determined. Chapter 4 is concerned with the case when an inhomogeneous plasma layer covers the aperture. The TE and TM solutions to the

scalar Helmholtz equations in each layer are transformed into voltages and currents, which are coupled together by boundary conditions across layers. The matrix elements and self-admittances follow the analysis of the model. They are reduced to those in Chapter 3 when the plasma layer is reduced to free space. Numerical calculation is discussed in Chapter 5. Chapter 6 presents the conclusion of the thesis.

Chapter 2

MODAL ANALYSIS

2.1 Fundamentals of Cylindrical Electromagnetic Wave Theory

In a simple, nonconducting source-free medium where the electric charge density ρ, the current density J and the electrical conductivity σ are all zeroes, one can derive the following wave equations from Maxwell's equations [9] :

$$\nabla^2\vec{\mathcal{E}} - \mu\epsilon\partial^2\vec{\mathcal{E}}/\partial t^2 = 0 \ , \tag{2.1}$$

$$\nabla^2\vec{\mathcal{H}} - \mu\epsilon\partial^2\vec{\mathcal{H}}/\partial t^2 = 0 \ . \tag{2.2}$$

where $\vec{\mathcal{E}} = \vec{\mathcal{E}}(\vec{r},t)$ and $\vec{\mathcal{H}} = \vec{\mathcal{H}}(\vec{r},t)$ are respectively the instantaneous values of the electric and magnetic field intensities in a linear, isotropic and homogeneous medium characterized by the permittivity ϵ and permeability μ .

For time-harmonic fields, vector phasors $\vec{E} = \vec{E}(\vec{r})$ and $\vec{H} = \vec{H}(\vec{r})$ are used to denote the space part of $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ as follows :

$$\vec{\mathcal{E}} = \mathrm{Re}[\vec{E}(\vec{r})e^{j\omega t}] = \mathrm{Re}(\vec{E}e^{j\omega t}) \ , \tag{2.3}$$

$$\vec{\mathcal{H}} = \mathrm{Re}[\vec{H}(\vec{r})e^{j\omega t}] = \mathrm{Re}(\vec{H}e^{j\omega t}) \ , \tag{2.4}$$

where Re means " the real part of " and ω is the oeprating angular frequency.

The corresponding vector Helmholtz's equations for $\vec{E}$ and $\vec{H}$ can also be derived from time-harmonic Maxwell's equations and are given by

$$\nabla^2\vec{E} + \omega^2\mu\epsilon\vec{E} = 0 , \quad (2.5)$$

$$\nabla^2\vec{H} + \omega^2\mu\epsilon\vec{H} = 0 . \quad (2.6)$$

For a cylindrical coordinate system, the three-dimensional Laplacian operator ∇^2 can be divided into two parts : $\nabla_{r\phi}^2$ for the transversal coordinates and ∇_z^2 for the longitudinal coordinate along which waves propagate. Thus

$$\nabla^2 = \nabla_{r\phi}^2 + \nabla_z^2 , \quad (2.7)$$

where

$$\nabla_{r\phi}^2 = [\partial(r\partial/\partial r)/\partial r]/r + (\partial^2/\partial\phi^2)/r^2 , \quad (2.8)$$

$$\nabla_z^2 = \partial^2/\partial z^2 . \quad (2.9)$$

To begin the analysis, one assumes a lossless medium in the +z- direction such that waves vary sinusoidally in that direction with a propagation constant β, or equivalently k_z. Thus

$$\vec{E} = \vec{E}^0(r,\phi)e^{-jk_z z} = \vec{E}^0 e^{-jk_z z} , \quad (2.10)$$

$$\vec{H} = \vec{H}^0(r,\phi)e^{-jk_z z} = \vec{H}^0 e^{-jk_z z} \quad (2.11)$$

where $\vec{E}^{o} = \vec{E}^{o}(r,\phi)$ and $\vec{H}^{o} = \vec{H}^{o}(r,\phi)$ are two- dimensional vector phasors that depend on transversal coordinates.

$\vec{E}^{o}$ and $\vec{H}^{o}$ can be decomposed into longitudinal and transversal components as

$$\vec{E}^{o} = \hat{a}_z E_z^o + \vec{E}_t^o \;, \tag{2.12}$$

$$\vec{H}^{o} = \hat{a}_z H_z^o + \vec{H}_t^o \;, \tag{2.13}$$

or can be decomposed further in terms of coordinate components as

$$\vec{E}^{o} = \hat{a}_z E_z^o + \hat{a}_r E_r^o + \hat{a}_\phi E_\phi^o \;, \tag{2.14}$$

$$\vec{H}^{o} = \hat{a}_z H_z^o + \hat{a}_r H_r^o + \hat{a}_\phi H_\phi^o \;. \tag{2.15}$$

Maxwell's equations for time-harmonic fields are

$$\nabla \times \vec{E} = -j\omega\mu \vec{H} \;, \tag{2.16}$$

$$\nabla \times \vec{H} = j\omega\epsilon \vec{E} \;. \tag{2.17}$$

Applying Equations (2.10,2.11,2.14,2.15) to Equations (2.16,2.17) and equating corresponding terms on both sides of the resulting equations, one obtains

$$(\partial E_z^o/\partial \phi)/r + jk_z E_\phi^o = -j\omega\mu H_r^o \;, \tag{2.18}$$

$$jk_z E_r^o + \partial E_z^o/\partial r = j\omega\mu H_\phi^o \;, \tag{2.19}$$

$$[\partial(rE_\phi^o)/\partial r - \partial E_r^o/\partial \phi]/r = -j\omega\mu H_z^o \;, \tag{2.20}$$

$$(\partial H_z^o/\partial \phi)/r + jk_z H_\phi^o = j\omega\epsilon E_r^o , \qquad (2.21)$$

$$jk_z H_r^o + \partial H_z^o/\partial r = -j\omega\epsilon E_\phi^o , \qquad (2.22)$$

$$[\partial(rH_\phi^o)/\partial r - \partial H_r^o/\partial \phi]/r = j\omega\epsilon E_z^o . \qquad (2.23)$$

Solving for transversal components E_r^o, E_ϕ^o, H_r^o and H_ϕ^o in terms of longitudinal components E_z^o and H_z^o from Equations (2.18) through (2.23) gives

$$E_r^o = -j[k_z(\partial E_z^o/\partial r) + \omega\mu(\partial H_z^o/\partial\phi)/r]/(\omega^2\mu\epsilon - k_z^2) , \qquad (2.24)$$

$$E_\phi^o = -j[k_z(\partial E_z^o/\partial\phi)/r - \omega\mu(\partial H_z^o/\partial r)]/(\omega^2\mu\epsilon - k_z^2) , \qquad (2.25)$$

$$H_r^o = -j[k_z(\partial H_z^o/\partial r) - \omega\epsilon(\partial E_z^o/\partial\phi)/r]/(\omega^2\mu\epsilon - k_z^2) , \qquad (2.26)$$

$$H_\phi^o = -j[k_z(\partial H_z^o/\partial\phi)/r + \omega\epsilon(\partial E_z^o/\partial r)]/(\omega^2\mu\epsilon - k_z^2) . \qquad (2.27)$$

Hence if E^o and H^o can be determined, all other four components follow immediately from Equations (2.24) through (2.27).

By using Equations (2.7,2.10,2.11), one may reduce Equations (2.5,2.6) to

$$\nabla_{r\phi}^2 \vec{E}^o + (\omega^2\mu\epsilon - k_z^2)\vec{E}^o = 0 , \qquad (2.28)$$

$$\nabla_{r\phi}^2 \vec{H}^o + (\omega^2\mu\epsilon - k_z^2)\vec{H}^o = 0 . \qquad (2.29)$$

Substituting Equations (2.12,2.13) into Equations (2.28,2.29) reduces them to

$$\hat{a}_z[\nabla_{r\phi}^2 E_z^o + (\omega^2\mu\epsilon - k_z^2)E_z^o] + [\nabla_{r\phi}^2 + (\omega^2\mu\epsilon - k_z^2)]\vec{E}_t^o = 0 , \tag{2.30}$$

$$\hat{a}_z[\nabla_{r\phi}^2 H_z^o + (\omega^2\mu\epsilon - k_z^2)H_z^o] + [\nabla_{r\phi}^2 + (\omega^2\mu\epsilon - k_z^2)]\vec{H}_t^o = 0 . \tag{2.31}$$

Since the two terms in Equation (2.30) are independent of each other, they both vanish. Thus one has

$$\nabla_{r\phi}^2 E_z^o + (\omega^2\mu\epsilon - k_z^2)E_z^o = 0 . \tag{2.32}$$

Similar equation follows from Equation (2.31) as

$$\nabla_{r\phi}^2 H_z^o + (\omega^2\mu\epsilon - k_z^2)H_z^o = 0 . \tag{2.33}$$

Hence if E_z^o and H_z^o, subject to boundary conditions, can be solved for, one can then obtain $\vec{E}^o$ and $\vec{H}^o$ by using Equations (2.24) through (2.27) and Equations (2.14,2.15). $\vec{E}$ and $\vec{H}$ follow from Equations (2.10,2.11), and finally $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$ from Equations (2.3,2.4). This concludes the solutions to the source-free wave Equations (2.1,2.2).

2.2 Eigenfunctions of the Scalar Helmholtz's Equations

Equations (2.32,2.33) have the same mathematical form, and therefore identical general solutions. To solve for either of them, the technique of separation of variables is used to reduce the partial differential equation to two ordinary differential equations. Thus by letting

$$E^0 = R(r)\Phi(\phi) \quad , \tag{2.34}$$

after expanding and rearranging, Equation (2.32) becomes

$$r^2(d^2R/dr^2)/R + r(dR/dr)/R + (\omega^2\mu\epsilon - k_z^2)r^2 = -(d^2\Phi/d\phi^2)/\Phi \ . \tag{2.35}$$

If Equation (2.35) holds for all values of r and ϕ, both sides must equal a constant. Let this constant be m^2, Equation (2.35) is reduced to

$$d^2\Phi/d\phi^2 = -m^2\Phi \tag{2.36}$$

$$r^2(d^2R/dr^2)/R + r(dR/dr)/R + (\omega^2\mu\epsilon - k_z^2)r^2 - m^2 = 0 \ . \tag{2.37}$$

The solution of Equation (2.36) is

$$\Phi = e^{-jm\phi} \quad , \tag{2.38}$$

where m is a positive or negative integer.

To simplify notation, one lets

$$k_c^2 = \omega^2\mu\epsilon - k_z^2 \ . \tag{2.39}$$

Substituting Equation (2.39) into Equation (2.37) and rearranging gives

$$d^2R/dr^2 + (dR/dr)/r + (k_c^2 - m^2/r^2)R = 0 \ . \tag{2.40}$$

Equation (2.40) is the Bessel equation of order m, the general solution of which is

$$R = A_m(k_z)J_m(k_c r) + B_m(k_z)N_m(k_c r) \quad , \qquad (2.41)$$

where $J_m(k_c r)$ and $N_m(k_c r)$ are respectively the Bessel functions of the first and second kind of order m; $A_m(k_z)$ and $B_m(k_z)$ are constants for fixed values of m and k_z.

The solution to Equation (2.40) can also be expressed in terms of the linear combinations of J_m and N_m as

$$R = A_m(k_z)H_m^{(1)}(k_c r) + B_m(k_z)H_m^{(2)}(k_c r) \qquad (2.42)$$

where

$$H_m^{(1)}(k_c r) = J_m(k_c r) + jN_m(k_c r) \quad , \qquad (2.43)$$

$$H_m^{(2)}(k_c r) = J_m(k_c r) - jN_m(k_c r) \quad . \qquad (2.44)$$

$H_m^{(1)}(k_c r)$ and $H_m^{(2)}(k_c r)$ are called the Hankel functions of order m, first and second kind, respectively. $H_m^{(1)}$ describes radially inward traveling waves while $H_m^{(2)}$ represents waves traveling outward.

If the permittivity ϵ is complex, which occurs in a conducting medium as in the plasma, one knows from Equation (2.39) that k_c will also become complex, the solution to Equation (2.40) is given by Equations (2.41,2.42) with complex arguments.

In conclusion, the solution to Equation (2.32) is obtained by using Equations (2.34,2.38,2.42) as

$$E_z^o = [A_m(k_z)H_m^{(1)}(k_c r) + B_m(k_z)H_m^{(2)}(k_c r)]e^{-jm\phi} \quad . \qquad (2.45)$$

A similar solution to Equation (2.33) is

$$H_z^0 = [C_m(k_z)H_m^{(1)}(k_c r) + D_m(k_z)H_m^{(2)}(k_c r)]e^{-jm\phi} \quad . \quad (2.46)$$

Other four components can be obtained from Equations (2.24) through (2.27). Multiplying each of these solutions by $e^{-jk_z z}$ will give the eigenfunction to the corresponding scalar Helmholtz's equation in Equations (2.5,2.6). Among them the axial eigenfunctions are

$$E_{z;m,k_z} = [A_m(k_z)H_m^{(1)}(k_c r) + B_m(k_z)H_m^{(2)}(k_c r)]e^{-j(m\phi+k_z z)} \quad , \quad (2.47)$$

$$H_{z;m,k_z} = [C_m(k_z)H_m^{(1)}(k_c r) + D_m(k_z)H_m^{(2)}(k_c r)]e^{-j(m\phi+k_z z)} \quad . \quad (2.48)$$

2.3 External Fields in terms of Given Aperture Fields.

For a rectangular slot antenna on the body of a cylindrical vehicle with its axis pointing in the +z-direction and a radius d, two field components $E_z(d,\phi,z)$ and $E_\phi(d,\phi,z)$ at the aperture are assumed known, it is desired to express six field components external to the aperture in terms of these two.

For waves traveling radially outward from the aperture, one uses only $H_m^{(2)}$ in Equations (2.47,2.48). Thus

$$E_{z;m,k_z}(r,\phi,z) = \mathcal{E}_{z;m}(k_z)H_m^{(2)}(k_c r)e^{-j(m\phi+k_z z)} \quad , \quad (2.49)$$

$$H_{z;m,k_z}(r,\phi,z) = \mathcal{H}_{z;m}(k_z)H_m^{(2)}(k_c r)e^{-j(m\phi+k_z z)} \quad , \quad (2.50)$$

where $r > d$ and $\mathcal{E}_{z;m}(k_z)$ and $\mathcal{H}_{z;m}(k_z)$ are expansion coefficients to be determined.

The general solution is a superposition of eigenfunctions of different values of m and k_z. Thus

$$
\begin{aligned}
E_z(r,\phi,z) &= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} E_{z;m,k_z}(r,\phi,z)\, dk_z \\
&= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{E}_{z;m}(k_z)\, H_m^{(2)}(k_c r)\, e^{-j(m\phi+k_z z)}\, dk_z \,, \quad (2.51)
\end{aligned}
$$

$$
\begin{aligned}
H_z(r,\phi,z) &= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} H_{z;m,k_z}(r,\phi,z)\, dk_z \\
&= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{H}_{z;m}(k_z)\, H_m^{(2)}(k_c r)\, e^{-j(m\phi+k_z z)}\, dk_z \,. \quad (2.52)
\end{aligned}
$$

Multiplying Equations (2.24) through (2.27) by $e^{-jk_z z}$, from Equations (2.10,2.11,2.39), one obtains the following eigenfunctions for other four components :

$$
\begin{aligned}
E_{r;m,k_z}(r,\phi,z) &= E_r^o e^{-jk_z z} \\
&= -(j/k_c^2)\left[k_z\, (\partial E_{z;m,k_z}/\partial r) \right. \\
&\quad \left. + (\omega\mu/r)\, (\partial H_{z;m,k_z}/\partial \phi) \right] , \quad (2.53)
\end{aligned}
$$

$$
\begin{aligned}
E_{\phi;m,k_z}(r,\phi,z) &= E_\phi^o e^{-jk_z z} \\
&= -(j/k_c^2)\left[(k_z/r)\, (\partial E_{z;m,k_z}/\partial \phi) \right. \\
&\quad \left. - \omega\mu\, (\partial H_{z;m,k_z}/\partial r) \right] , \quad (2.54)
\end{aligned}
$$

$$
\begin{aligned}
H_{r;m,k_z}(r,\phi,z) &= H_r^o e^{-jk_z z} \\
&= -(j/k_c^2)\left[k_z\, (\partial H_{z;m,k_z}/\partial r) \right.
\end{aligned}
$$

$$- (\omega\epsilon/r)(\partial E_{z;m,k_z}/\partial \phi)] \, , \qquad (2.55)$$

$$H_{\phi;m,k_z}(r,\phi,z) = H_\phi^0 e^{-jk_z z}$$

$$= -(j/k_c^2)[(k_z/r)(\partial H_{z;m,k_z}/\partial \phi)$$

$$+ \omega\epsilon(\partial E_{z;m,k_z}/\partial r)] \, . \qquad (2.56)$$

Substituting Equations (2.51,2.52) into Equations (2.53) through (2.56) and taking the derivatives yields

$$E_{r;m,k_z}(r,\phi,z) = -(j/k_c^2)[k_z k_c \mathcal{E}_{z;m}(k_z) H_m'^{(2)}(k_c r)$$

$$- jm\omega\mu\mathcal{H}_{z;m}(k_z) H_m^{(2)}(k_c r)/r]e^{-j(m\phi+k_z z)} \, ,$$

$$(2.57)$$

$$E_{\phi;m,k_z}(r,\phi,z) = -(j/k_c^2)[-jmk_z\mathcal{E}_{z;m}(k_z) H_m^{(2)}(k_c r)/r$$

$$-\omega\mu k_c\mathcal{H}_{z;m}(k_z) H_m'^{(2)}(k_c r)]e^{-j(m\phi+k_z z)} \, ,$$

$$(2.58)$$

$$H_{r;m,k_z}(r,\phi,z) = -(j/k_c^2)[k_z k_c\mathcal{H}_{z;m}(k_z) H_m'^{(2)}(k_c r)$$

$$+ jm\omega\epsilon\mathcal{E}_{z;m}(k_z) H_m^{(2)}(k_c r)/r]e^{-j(m\phi+k_z z)} \, ,$$

$$(2.59)$$

$$H_{\phi;m,k_z}(r,\phi,z) = -(j/k_c^2)^2[-jmk_z\mathcal{H}_{z;m}(k_z) H_m^{(2)}(k_c r)/r$$

$$+\omega\epsilon k_z\mathcal{E}_{z;m}(k_z) H_m'^{(2)}(k_c r)]e^{-j(m\phi+k_z z)} \, ,$$

$$(2.60)$$

where $H_m'^{(2)}(k_c r) = dH_m^{(2)}(k_c r)/d(k_c r)$.