RANKS AND BOUNDS FOR INDECOMPOSABLE MODULES OVER ONE-DIMENSIONAL NOETHERIAN RINGS

by

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A THESIS

Presented to the Faculty of
The Graduate College at the University of Nebraska
In Partial Fulfillment of Requirements
For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professor Sylvia Wiegand

Lincoln, Nebraska

December, 2007

UMI Number: 3283908

UMI Microform 3283908

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RANKS AND BOUNDS
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University of Nebraska, 2007

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We consider one-dimensional, reduced Noetherian rings R with finite normalization. We assume that there exists a positive integer N_R such that, for every indecomposable finitely generated torsion-free R-module M and for every minimal prime ideal P of R, the dimension of M_P, as a vector space over the field R_P, R localized at P, is less than or equal to N_R. We call the set of vector-space dimensions the *rank-set* of the module. We often call the sequence of vector-space dimensions the *rank* of the module. We are interested in what rank-sets occur for indecomposable R-modules.

In Chapter II, with Meral Arnavut and Sylvia Wiegand, many possible ranks are eliminated. In the constant rank case, that is, when we have a module M such that the vector-space dimension of M_P is the same for every minimal prime P of R, the only ranks occurring for indecomposable modules are $1, 2, 3, 4$ and 6. In the non-constant rank case, we show that for $n \geq 8$ there are no indecomposable modules with rank-sets between n and $2n-8$. On the other hand, for each $n \geq 8$, we construct an indecomposable module with rank-set the set of consecutive integers from n to $2n-7$.

In Chapter III, for each set of consecutive integers not ruled out in Chapter II, we produce a semilocal ring and an indecomposable module over that ring having that set as its rank-set. Could other ranks occur for indecomposable modules? To answer this, we construct some indecomposable modules with ranks of non-consecutive integers.

We then give some conditions to show some additional sets of non-consecutive integers never occur as the rank-sets of indecomposable modules.

In Chapter IV, with Nick Baeth, we prove the last case in a theorem listing the ranks that occur for indecomposable modules over a local ring. This result was previously known with an additional hypothesis on the characteristic of R and its residue field. This hypothesis was assumed in the work of Chapter II, but now we have eliminated it, and so, the results in Chapter II hold in more generality.

ACKNOWLEDGMENT

I would like to express my sincere appreciation to my thesis advisor Sylvia Wiegand for her encouragement, patience, advice, and for assistance with my thesis.

I would also like to thank my husband Bert Luckas, my parents Al and Pam Hendrick, my sister Amanda Line, and my in-laws Al and Mary Luckas for their support, encouragement and many hours of babysitting.

Thank you to the members of my supervisory committee: Tom Marley, Helen Moore, Judy Walker and Roger Wiegand. Thank you also to the mathematicians I have worked with: Meral Arnavut, Nicholas Baeth, Christina Eubanks-Turner and Serpil Saydam. Finally, thank you to Melissa Erdmann for being a mentor to me over the past few years.

Contents

Chapter I

Introduction

My thesis consists of three related papers, each occurring as a chapter here. Chapter II, co-authored with Meral Arnavut and Sylvia Wiegand, concerns ranks that occur for indecomposable modules over certain non-local rings. Chapter III gives more examples of indecomposable modules, and shows examples of other ranks that cannot occur for indecomposables. Chapter IV, with Nick Baeth, takes care of the remaining case of a theorem listing the ranks that occur for indecomposable modules over certain local rings; this makes the results of the first two papers hold in more generality.

Each paper contains its own introduction, giving the history of the problem, relevant definitions and previous results. Here we begin with some basic definitions which can be found in Matsumura [27] or Lang [25].

Basic ring definitions. A *Noetherian* ring, is a ring R that satisfies the ascending chain condition on ideals, that is, every ascending chain of ideals $I_1 \subset I_2 \subset \ldots$ must stop after a finite number of steps. An equivalent condition is that every ideal of R is finitely-generated. An *Artinian* ring is a ring that satisfies the descending chain condition on ideals, that is, every descending chain of prime ideals must stop. The *Krull dimension* of a ring is the maximal length of a chain of prime ideals. Thus a

one-dimensional ring, such as we consider here, has only minimal prime ideals and maximal ideals with no prime ideals in between.

A ring R is a *domain* if the ideal (0) is prime, equivalently, R has no non-zero zero-divisors. The *Jacobson radical*, which we often call just the *radical* of a ring, is the intersection of all maximal ideals. The *nilradical* of a ring is the intersection of all prime ideals of R, and is the set of all *nilpotent* elements, that is, the elements x such that $x^n = 0$ for some integer n. When there are no non-zero nilpotent elements, that is, the nilradical is zero, then we say the ring is *reduced*.

A *local* ring (R, m, k) is a commutative ring R with a unique maximal ideal m. In this case, the *residue field* of R is the field $k := R/m$. A *semilocal* ring has only finitely many maximal ideals. Let (R, m, k) be a local ring. When R has characteristic p for some prime p, then k does also. If instead, the characteristic of k is 0, then so is the characteristic of R. In either of these two cases, we say R is equicharacteristic. This is equivalent to R containing a field.

Basic module definitions. A module M over a ring R is *finitely generated* if $M = Rx_1 + \ldots + Rx_n$ for some integer n and $x_1, \ldots, x_n \in R$. We say M is *torsion-free* if whenever $rm = 0$ for $m \in M$, $m \neq 0$ and $r \in R$, then r is a zero divisor of R. A module is *indecomposable* if it cannot be written as a direct sum of two non-zero modules.

Let R be a ring, and M a finitely-generated R-module. An *M-regular* element of R is an element r such that $rx \neq 0$ for all non-zero $x \in M$. A sequence $x_1, \ldots, x_n$ in R is *M-regular* if

(1) x_1 is M-regular, x_2 is $M/(x_1)$-regular, $\ldots$, x_i is $M/(x_1, \ldots, x_{i-1})$-regular, and

(2)$M/(x_1, \ldots, x_n) \neq 0$.

For a local ring (R, m, k), the *depth* of a finitely-generated R-module M, $\operatorname{depth}(M)$, is the length of a maximal M-regular sequence in m. M is *Cohen Macaulay* if $M \neq 0$

and $\operatorname{depth}(M) = \dim(M)$. We say a ring R is *Cohen Macaulay* if it is a Cohen Macaulay module over itself. The *maximal Cohen Macaulay* modules are those modules for which $\operatorname{depth}(M) = \dim(R)$.

The *m-adic completion* of a local ring (R, m, k) is the completion defined by the topology $\{m^i\}_{i=1,2,\ldots}$, and is complete in that a Cauchy sequence has a unique limit.

Field extensions. Let $L \subseteq K$ be an extension of fields. An element $k \in K$ is called *algebraic* over L if k is the root of some polynomial over L. An element $k \in K$ is called *separable* over L if it is the root of a polynomial over L having no repeated roots. The extension $L \subseteq K$ is called algebraic (or separable) if every element in K is algebraic (separable) over L. A field L is said to be *perfect* if every algebraic extension of L is separable. An algebraic field extension K over L is *normal* if every irreducible polynomial over L with a root in K factors into linear factors over K. An algebraic extension which is both normal and separable is called *Galois*. Suppose $L \subseteq K$ is a field extension, and the characteristic of L is p for some prime integer p. An element $k \in K$ is said to be *purely inseparable* over L if $k^{p^n} \in L$ for some integer n. The extension K over L is called purely inseparable if every element in K is purely inseparable over L.

Chapter II

Indecomposable modules over one-dimensional Noetherian rings

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Abstract. In this article we consider finitely generated torsion-free modules over certain one-dimensional commutative Noetherian rings R. We assume there exists a positive integer N_R such that, for every indecomposable R-module M and for every minimal prime ideal P of R, the dimension of M_P, as a vector space over the field R_P, is less than or equal to N_R. If a nonzero indecomposable R-module M is such that all the localizations M_P as vector spaces over the fields R_P have the *same* dimension r, for every minimal prime P of R, then $r = 1, 2, 3, 4$ or 6. Let n be an integer ≥ 8. We

show that if M is an R-module such that the vector space dimensions of the M_P are between n and $2n-8$, then M decomposes non-trivially. For each $n \geq 8$, we exhibit a semilocal ring and an indecomposable module for which the relevant dimensions range from n to $2n-7$. These results require a mild equicharacteristic assumption; we also discuss bounds in the non-equicharacteristic case.

1 Introduction

Over the past forty years, a great deal of progress has been made towards determining *representation type* over certain Noetherian one-dimensional rings, that is, on the question: What are the isomorphism classes of the indecomposable modules? In the classical work from the sixties, the rings studied are one-dimensional domains that are finitely generated over the integers and are contained in algebraic number fields. See, for example, [14], [16], [18], [21], [22].

In this article we consider a larger class of one-dimensional rings for which, over the past twenty years, a concept of the size or *rank* of a module, as well as reasonable criteria so that the indecomposable modules have finite rank, have been developed [1], [4], [3], [9], [15], [19], [24], [23]. We study the question: For those rings such that a bound exists on the ranks of indecomposable modules, what are the bounds?

In order to describe the setting and the results of this paper, we start with some terminology. We also give some history, recent developments, and useful previous results.

Basic Definitions/Notation 1.1. A commutative ring R is *reduced* if $r^n \neq 0$, for every non-zero $r \in R$ and every positive integer n. A one-dimensional, reduced Noetherian ring R for which the integral closure $\widetilde{R}$ (or normalization) in its quotient ring is finitely generated as an R-module (equivalently as an R-algebra) is called a *ring-order*. A module M is *torsion-free* if $rm = 0$, for $0 \neq r \in R$ and $0 \neq m \in M$, implies that r is a zero-divisor of R.

Let R be a ring-order with minimal prime ideals $P_1, \ldots, P_m$. If M is a torsion-free R-module and i is an integer with $1 \leq i \leq m$, set $r_i := \dim_{R_{P_i}}(M_{P_i})$, the dimension of M_{P_i} as an R_{P_i}-vector space. The *rank* of M, or $\operatorname{rank}(M)$, is the m-tuple $(r_1, \ldots, r_m)$; we often write $\operatorname{Ranks}(M)$ for the set $\{r_1, \ldots, r_m\}$. If, for some integer

N, Ranks$(M) = \{N\}$, that is, $(r_1, \ldots, r_m) = (N, \ldots, N)$, then we say M has *constant rank* N and write rank$(M) = N$. For example, if R has a unique minimal prime ideal, then every module has constant rank. If $a \leq r_i \leq b$, for each $1 \leq i \leq m$, we write Ranks$(M) \subseteq \text{Int}\,[a, b]$; that is, $\text{Int}\,[a, b]$ represents the integers in the closed interval from a to b on the real line. We say R has *bounded representation type* if there exists an integer N_R such that, for each $i \in \mathcal{N}$ with $1 \leq i \leq m$ and for each indecomposable finitely generated torsion-free R-module M, $\dim_{R_{P_i}}(M_{P_i})$ is less than or equal to N_R. Thus, if R has bounded representation type and M is an indecomposable R-module, then Ranks$(M) \subseteq \text{Int}\,[0, N_R]$, for some positive integer N_R (ostensibly N_R depends upon R). We also refer to the *spread* of Ranks(M), the least integer s so that Ranks$(M) \subseteq \text{Int}[N, N + s]$, where N is the smallest rank entry of M.

Setting 1.2. For the remainder of this paper, unless otherwise stated, all rings are ring-orders and all modules are finitely generated and torsion-free. The setting is further restricted in (2.2).

Some history. In their 1988 paper, Levy and Haefner show that arbitrarily large ranks occur for indecomposable modules over ring-orders of bounded representation type; that is, for every n, there exist a ring-order R of bounded representation type and an indecomposable R-module M so that some rank entries of M are bigger than n [20]. The ring-orders R of the examples in [20] are such that $N_R = 1$ locally; that is, for each maximal ideal $\mathcal{M}$ of R, every torsion-free finitely generated $R_{\mathcal{M}}$-module is isomorphic to a direct sum of ideals. If R is a ring-order of bounded representation type for which $N_R = 1$ locally, however, a result of S. Wiegand shows that there are no indecomposable R-modules having rank 3 or more and small rank spread: For a ring-order R of bounded representation type satisfying the local $N_R = 1$ condition, if n is an integer ≥ 3 and M is a torsion-free R-module such that Ranks$(M) \subseteq \text{Int}\,[n, 2n - 2]$, then M decomposes [35, Theorem 3.3]. Moreover, for every positive integer n, there exist a semi-local ring-order R of bounded representation type with $N_R = 1$ locally and a finitely generated indecomposable torsion-free R-module M with Ranks$(M) \subseteq \text{Int}\,[n, 2n - 1]$ [35, Theorem 2.2].

Since then a succession of results have given steadily decreasing bounds on the ranks of indecomposable modules for local rings of bounded representation type and for the ranks of *constant* rank indecomposables globally. In their 1991 paper, Leo Chouinard and S. Wiegand establish a bound of $N = 39$ on the ranks of indecom-

posables of constant rank over all ring-orders of bounded representation type [10]. Roger Wiegand and S. Wiegand improve the constant rank bound to $N = 12$ in their 1994 article; they also establish a bound of 4 for local ring-orders of bounded representation type [34]. In his recent article [5], Nicholas Baeth has improved the local bound to $N = 3$ for complete local rings of bounded representation type satisfying the additional condition:

1.3. R contains a field not of characteristic 2, 3 or 5, and the residue field (mod the maximal ideal) is perfect.

It follows from Baeth's result that the local bound is $N = 3$ for non-complete local rings satisfying (1.3) (see Theorem 3.4 of Section 3). It then follows that the constant bound is 6 for ring-orders of bounded representation type with (1.3) that are not necessarily local nor complete and this is the best possible bound: A nontrivial constant rank indecomposable module has rank $1, 2, 3, 4$ or 6 by part (1)(i) of Main Theorem 1.4 below; there are examples of indecomposables with these constant ranks.

The main theorem of this article describes bounds on the ranks of indecomposable modules of non-constant rank over a ring-order R that satisfies condition (1.3) and has bounded representation type, provided that the rank entries are not too spread out. This is similar to the result of S. Wiegand cited above, but without the local $N_R = 1$ condition. For a module M to be indecomposable with one rank entry $n \geq 8$, the rank-tuple has to have a spread of more than $\frac{n-7}{2}$; see Corollary 4.2. We also prove the other direction: There exist indecomposable modules having rank spread barely bigger than the spread that implies the modules decompose.

Theorem 1.4. **(1)** *Let R be a ring-order of bounded representation type such that R is equicharacteristic, not of characteristic* 2, 3 *or,* 5, *with perfect residue fields, and let M be an R-module.*

(i) *If M is indecomposable and* $\mathrm{Ranks}(M) = \{r\}$, *then* $r = 1, 2, 3, 4,$ *or* 6.

(ii) *If there exists a positive integer $n \geq 8$ such that* $\mathrm{Ranks}(M) \subseteq \mathrm{Int}[n, 2n-8]$, *then M decomposes non-trivially.* (*It has a direct summand of constant rank* 6.)

(2) *For each $n \geq 8$, there exists a semilocal ring-order of bounded representation type and an indecomposable module M such that* $\mathrm{rank}(M) = (n, n+1, \ldots, 2n-7)$.

Corollary 1.5. *Suppose that R is a ring-order as described in Main Theorem* 1.4(1). *If M is an indecomposable module such that the spread of the ranks is $\leq i$, for some integer i, then* $\mathrm{Ranks}(M) \subseteq \mathrm{Int}[0, 7+2i]$.

Proof. Let N and s denote the smallest rank entry and the spread of M, respectively, so that $s \leq i$ and $N+s$ is the largest rank entry. First suppose that $N \geq 8$. If $N+s \leq N+i < 2N-7$, then M would decompose non-trivially; thus

$$N+i \geq 2N-7 \implies N \leq 7+i \implies N+s \leq 7+2i,$$

as desired. If $N \leq 7$, then we have $N+s \leq N+i \leq 7+2i$ again, and the proof is complete. □

Table 1.6. ***Possible rank sets.*** *Here is a table showing the possible rank sets for a nonzero indecomposable module over a ring-order, as described in the Main Theorem* 1.4*, using part (1) and Corollary* 1.5*:*

spread 0 (constant rank): $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{6\}$.
spread 1: $\{0,1\}$, $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\{4,5\}$, $\{5,6\}$, $\{6,7\}$, $\{7,8\}$, $\{8,9\}$.
spread 2: $\{0,1,2\}$, $\{1,2,3\}$, $\{2,3,4\}$, $\{3,4,5\}$, $\ldots$, $\{9,10,11\}$.
spread 3: $\{0,1,2,3\}$, $\{1,2,3,4\}$, $\{2,3,4,5\}$, $\ldots$, $\{10,11,12,13\}$.
$\vdots$ $\vdots$
spread i: $\{0,1,\ldots,i\}$, $\{1,2,\ldots,1+i\}$, . . ., $\{7+i,8+i,\ldots,7+2i\}$.
$\vdots$ $\vdots$

Question 1.7. Do all of these ranks occur?

We believe so, and we are writing the proof, to appear separately. The last rank set listed for each spread, from spread 1 on, exists by part (2) of Theorem 1.4. It is straightforward for the reader to check that constant rank 1, 2, 3, 4 and 6 occur; examples are given in [34].

Remarks on the non-equicharacteristic case. Prior to Baeth's result, the Wiegands and Çimen showed that the ranks possible for finitely generated indecomposable torsion-free modules over a local ring of bounded representation type are among a finite list, bounded by 4 [13, Theorem 4.6]. The list, given in Theorem 6.3 below, does not require the equicharacteristic condition (1.3); M. Arnavut and S. Wiegand used Theorem 6.3 to prove the following: [1]

[1]The proof of Theorem 1.8 appears in Meral Arnavut's thesis [1].

Theorem 1.8. *[1] If R is a ring-order of bounded representation type (not assuming condition (1.3)), n is an integer greater than or equal to 18, and M is an R-module with* $\mathrm{Ranks}(M) \subseteq \mathrm{Int}[n, 2n-14]$*, then M decomposes non-trivially. Such a module M has a direct summand of constant rank* 12.

We include in Section 6 a brief discussion of Theorem 1.8 (called Theorem 6.1) and we make a few additional remarks about the status of the non-equicharacteristic case. We do not provide the proof here; the proof is computational like the proof of our main theorem, but even messier. We expect the bounds of Main Theorem 1.4 hold for the general case. To reprove Baeth's theorem for the general case, however, probably requires very technical matrix computations.

In Section 2 we give additional terminology and relevant previous results. In Section 3 we set up a matrix representation of the rank problem and prove some computational technical lemmas. In Section 4 we prove part (1) of the Main Theorem 1.4; in Section 5 we prove part (2).

2 Terminology and background

In this section we give notation and results needed for later sections.

Notation 2.1. For an arbitrary local ring R with maximal ideal $\mathcal{M}$, we denote by $\widehat{R}$ the completion of R with respect to the $\mathcal{M}$-adic topology; for M an R-module, $\widehat{M}$ denotes the $\widehat{R}$-module $M \otimes_R \widehat{R}$. An $\widehat{R}$-module N is *extended from an R-module* if there exists an R-module M such that $N = \widehat{M}$.

Now let R be a ring-order (defined in (1.1)). A maximal ideal $\mathcal{M}$ of R is *singular* if $R_{\mathcal{M}}$ is neither a field nor a discrete valuation ring. The ring $S^{-1}R$, where S is the complement in R of the union of the singular maximal ideals, is called the *singular semilocalization* of R, and denoted by R_{sing}. The singular maximal ideals are those that contain $(R : \widetilde{R}) = \{a \in R \,|\, a\widetilde{R} \subseteq R\}$, the conductor of R. Thus R_{sing} is a semilocal ring-order.

In 2.2 we establish the setting for most of the paper; often in later sections we add the conditions of (2.6.1). All modules are finitely generated and torsion-free, as required in 1.2.

Setting 2.2. Let R be a ring-order such that

(1) R is connected (that is, R is not isomorphic to a direct product of two non-trivial rings), and $R \neq \widetilde{R}$, the integral closure of R in its total quotient ring;

(2) Every minimal prime ideal of R is contained in a singular maximal ideal;

(3) Every zero-divisor of R is contained in the union of all singular maximal ideals; and

(4) $R \subseteq R_{\text{sing}} \subsetneq K$, where K is the total quotient ring of R.

By the following theorem, there is no loss of generality in assuming that $R = R_{sing}$, so that R is semilocal, for the analysis of bounds of indecomposables.

Theorem 2.3. *[35, Theorems 1.3 and 1.4] Let R, S and R_{sing} be as in* 1.1 *and* 2.2. *Then R has bounded representation type if and only if R_{sing} has bounded representation type. Moreover, if A is a torsion-free R-module and the R_{sing}-module $S^{-1}A$ has an R_{sing}-module B as a direct summand, then A has a direct summand C such that $S^{-1}C = B$.*

In the proof of Main Theorem 1.4 we consider all possible ranks for indecomposable modules over local ring-orders and then use the following local-global theorems (Theorem 2.4 and 2.5) to obtain and justify all possible ranks for indecomposable modules globally. Theorem 2.4 is used in Section 5 to construct indecomposable modules of specified ranks by gluing together modules defined at the localizations.

Theorem 2.4. *[30, Lemma 1.11] Let R be a semilocal ring-order with maximal ideals $\mathcal{M}_1, \ldots, \mathcal{M}_n$, and let A_i be a torsion-free $R_{\mathcal{M}_i}$-module, for each i with $1 \leq i \leq n$. For each pair i, j with $1 \leq i < j \leq n$ and for each minimal prime ideal P contained in $\mathcal{M}_i \cap \mathcal{M}_j$, we assume that $(A_i)_P \cong (A_j)_P$. (That is, $(A_i)_P$ and $(A_j)_P$ have the same vector space dimensions over R_P.) Then there exists a torsion-free R-module A (unique up to isomorphism) such that $A_{\mathcal{M}_i} \cong A_i$, for each $i \in I$.*

Theorem 2.5 is used in Section 4 to split off direct summands of constant rank.

Theorem 2.5. *[34, Theorem 4.8] Let R be a ring-order, let A be a torsion-free R-module, and let n be a positive integer. Suppose, for every maximal ideal $\mathcal{M}$ of R, $A_{\mathcal{M}}$ has a direct summand $X(\mathcal{M})$ of constant rank n. Then A has a direct summand X of constant rank n and $X_{\mathcal{M}} \cong X(\mathcal{M})$.*

Remarks 2.6.

(1) A a semilocal ring-order R has bounded representation type if and only if R has *finite representation type* by [13, Theorem 1.6.5] and [30, Theorem 1.9]; that is, R has finitely many indecomposable modules up to isomorphism.

Therefore, for the proof of the first part of the Main Theorem 1.4 in Section 4, we assume:

(2.6.1) R is as in 2.2, $R = R_{sing}$ (i.e. R is semilocal) and R has finite representation type, and R is equicharacteristic, but not of characteristic 2, 3 or 5, with perfect residue field.

(2) One of the main tools we use for our results is the list of possible ranks for indecomposable modules over local ring-orders of bounded (finite) representation type. In their 1994 article, R. Wiegand and S. Wiegand give such a list; also they describe an "indecomposable" module over a local ring-order of finite representation type having rank 4 [34]. Recently Nicholas Baeth has shown, however, that their supposed rank 4 indecomposable is not indecomposable, and further that 4 never occurs as a rank of an indecomposable module over a local ring-order of bounded representation type, if the ring is equicharacteristic but not of characteristic 2, 3 or 5 and has perfect residue field. We suspect that rank 4 never occurs even without the added restriction.

The theorem below, due to Baeth, gives all possible ranks for indecomposable modules over complete local ring-orders satisfying the equicharacteristic condition. It follows, see Theorem 3.4, that these are only possible ranks for indecomposables over non-complete local ring-orders.

Theorem 2.7. *[5, Theorem 4.2] Suppose that R is a complete local ring-order with finite representation type such that R is equicharacteristic, not of characteristic 2, 3 or 5, with perfect residue field. Let M be a non-zero indecomposable torsion-free R-module. Then R has at most three minimal prime ideals and:*

(2.7.1) *If R has one minimal prime ideal, then* $\operatorname{rank}(M) = 1, 2$ *or* 3.

(2.7.2) *If R has two minimal prime ideals, then* $\operatorname{rank}(M)$ *is one of the following:*

$$(1,0),(0,1),(1,1),(1,2),(2,1),(2,2).$$

(2.7.3) *If R has three minimal prime ideals then* $\operatorname{rank}(M)$ *is one of these:*

$$(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1),(1,1,2).$$

The fact that the ring-order R in Theorem 2.7 has at most three minimal prime ideals is due to Dade [14, page 410], [30, 2.9], [18, 4.5].

The following gluing theorem for rings is used to produce semilocal ring-orders with specified localizations for the examples in the second part of the Main Theorem 1.4.

Theorem 2.8. *[34, Theorem 4.6] Let k be an infinite field, and t an indeterminate over k. Let $(R_1, \mathcal{M}_1), \ldots, (R_n, \mathcal{M}_n)$ be reduced local k-algebras of dimension one, such that, for each i and for each minimal prime P of R_i, $(R_i)_P$ is k-isomorphic to $K(t)$, where K is some algebraic extension field of k, possibly depending on i and P. Let X be a finite one-dimensional partially ordered set with exactly n maximal elements $x_1, \ldots, x_n$, all of them non-minimal. Assume*

1. *For each i, there is an order-embedding $\phi_i : \operatorname{Spec} R_i \to X$ taking $\mathcal{M}_i$ to x_i; that is, the number of elements of X that are $\leq x_i$ is equal to the cardinality of $\operatorname{Spec}(R_i)$.*

2. *If $P \in \operatorname{Spec} R_i$ and $Q \in \operatorname{Spec} R_j$ are such that $\phi_i(P) = \phi_j(Q)$, then $(R_i)_P \cong (R_j)_Q$ as k-algebras.*

Then there is a semilocal k-algebra R with maximal ideals $\{\mathcal{N}_i \mid i = 1, \ldots, n\}$ such that $\operatorname{Spec} R$ is order-isomorphic to X (with $\mathcal{N}_i$ mapping to x_i) and $R_{\mathcal{N}_i} \cong R_i$, for all i.

The standard version of the Krull-Remak-Schmidt Theorem for a quasilocal ring (that is, for a ring E not necessarily commutative such that E modulo its Jacobson radical is Artinian) is the following:

Theorem 2.9. *[29, p. 79, Theorem 2.8] Let $\bigoplus_{i=1}^{m} M_i \cong \bigoplus_{j=1}^{n} N_j$, where the M_i and N_j are indecomposable modules over a quasilocal ring R. If End(M_i) is quasilocal, for each i, then $m = n$ and $M_i \cong N_i$ after renumbering.*

Unfortunately the uniqueness of decompositions of modules into indecomposable modules as given in the Krull-Remak-Schmidt Theorem 2.9 does not always hold, even when the ring is a local ring-order (see [34, Example 3.3]). The precise form of the Krull-Remak-Schmidt Theorem that is needed is Corollary 2.11(2), which follows from Proposition 2.10. See also [26, Theorem 3.4] for a proof of a more general version of Proposition 2.10, where R is any one-dimensional local ring. We use terminology established in 2.1.

Proposition 2.10. *[17, Lemma 2.5] Let R be a local ring-order, and suppose that M is a finitely generated torsion-free $\widehat{R}$-module. Then M is extended from an R-module if and only if $\dim_{\widehat{R}_p}(M_p) = \dim_{\widehat{R}_q}(M_q)$, for every pair p, q of minimal prime ideals of $\widehat{R}$ such that $p \cap R = q \cap R$ (that is, whenever p and q lie in the same fiber over* $\operatorname{minSpec}(R)$*). In particular, if M and N are $\widehat{R}$-modules with the same rank, then M is extended if and only if N is extended.*

Corollary 2.11. *[Roger Wiegand] Let R be a one-dimensional local ring with reduced completion $\widehat{R}$. If the map $\operatorname{Spec}(\widehat{R}) \to \operatorname{Spec}(R)$ is one-to-one (equivalently, the number of minimal primes of R is the same as the number of minimal primes of $\widehat{R}$), then*

1. *The map $M \mapsto \widehat{M}$, from isomorphism classes of R-modules to isomorphism classes of $\widehat{R}$-modules, is an isomorphism of monoids, and*

2. *The Krull-Remak-Schmidt Theorem holds for direct-sum decompositions of finitely generated R-modules.*

3 Matrix notation, non-complete rings, computations

This section contains various mundane technical details of the paper. The first project of the section is the introduction of efficient matrix notation so that results and proofs can be written more concisely. Second, we expand Baeth's Theorem 2.7 to non-complete local ring-orders in Theorem 3.4. Third, we prove lemmas and propositions needed for the proof of Theorem 4.1.

For establishing the matrix notation, let R be a ring-order, $\mathcal{M}$ a maximal ideal of R, and M a finitely generated torsion-free R-module such that $M_{\mathcal{M}}$ can be expressed as a direct sum of indecomposable $R_{\mathcal{M}}$-modules with ranks given in Theorem 2.7. Then, for the cases where $R_{\mathcal{M}}$ has either one, two or three minimal primes, we describe this decomposition using a matrix of non-negative integers as in 3.1, 3.2 or 3.3 below.

Notation 3.1. Matrix representation: the one minimal prime case: Suppose $\mathcal{M}$ is a maximal ideal of R containing exactly one minimal prime ideal P. Suppose also that the rank of every indecomposable $R_{\mathcal{M}}$-module appears on the list in (2.7.1).

Let I_1 denote a generic indecomposable $R_{\mathcal{M}}$-module of rank 1; that is, I_1 denotes an indecomposable $R_{\mathcal{M}}$-module X such that X_P has dimension one as an R_P-vector space. Similarly I_2 and I_3, represent generic indecomposable $R_{\mathcal{M}}$-modules having ranks 2 and 3, respectively, as displayed in the subscripts, and these are all. That is, the entire list of nontrivial generic indecomposables in the one minimal prime case is

$$I_1,\ I_2,\ I_3.$$

In assigning these subscripts we do not care if there are actually two non-isomorphic indecomposable modules of a given rank represented in the same way; for example, I_1 may stand for two or more non-isomorphic indecomposable $R_{\mathcal{M}}$-modules of rank 1.

With the assumptions above, let M be a finitely generated R-module. To the $R_{\mathcal{M}}$-module $M_{\mathcal{M}}$, we associate a three-tuple $\vec{w} = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \in \mathcal{N}^3$ to indicate that $M_{\mathcal{M}}$ is a direct sum of generic indecomposable $R_{\mathcal{M}}$-modules of form:

$$M_{\mathcal{M}} \text{``}=\text{''}\ I_1^{w_1} \oplus I_2^{w_2} \oplus I_3^{w_3}. \tag{3.1a}$$

The interpretation of this representation $\vec{w}$ for $M_{\mathcal{M}}$ is that $M_{\mathcal{M}}$ is a direct sum of indecomposable $R_{\mathcal{M}}$-modules, with w_1 indecomposables of rank 1, w_2 indecomposables of rank 2, and w_3 indecomposables of rank 3. As mentioned above, it does not matter that indecomposables having the same index may not be isomorphic. The vector $\vec{w}$ is not necessarily unique for M, unless the Krull-Remak-Schmidt Theorem holds for $R_{\mathcal{M}}$.

Now suppose that $M_{\mathcal{M}}$, represented as in the expression above, has $\operatorname{rank}(M_{\mathcal{M}}) = r$, for $r \in \mathcal{N}$, i.e. $\dim_{R_P}(M_P) = r$, and $\vec{w} := \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}$, for some $w_1, w_2, w_3 \in \mathcal{N}$. Then we have

$$r = \operatorname{rank}(M_{\mathcal{M}}) = \operatorname{rank}(I_1^{w_1} \oplus I_2^{w_2} \oplus I_3^{w_3}) = 1 \cdot w_1 + 2 \cdot w_2 + 3 \cdot w_3, \tag{3.1b}$$

$$r = w_1 + 2w_2 + 3w_3. \tag{3.1c}$$

Notation 3.2. Matrix representation: the two minimal prime case: Suppose $\mathcal{M}$ is a maximal ideal of R containing exactly two minimal primes P_1 and P_2. Suppose also that every indecomposable $R_{\mathcal{M}}$-module has its rank-tuple one of the list in (2.7.2).

In analogy to the one minimal prime case, we represent nontrivial indecomposable $R_{\mathcal{M}}$-modules by the letter I with subscripts that display the rank-tuples. Thus

$$I_{(10)},\ I_{(01)},\ I_{(11)},\ I_{(12)},\ I_{(21)},\ I_{(22)}$$

are the entire list of nontrivial generic indecomposable $R_{\mathcal{M}}$-modules, and they have the rank-tuples displayed in the subscripts. For example, $I_{(10)}$ denotes a generic indecomposable $R_{\mathcal{M}}$-module of rank $(1,0)$, that is, an indecomposable $R_{\mathcal{M}}$-module X such that X_{P_1} has dimension one as an R_{P_1}-vector space, and X_{P_2} has dimension zero as an R_{P_2}-vector space. In assigning these subscripts we do not care whether there might actually be two non-isomorphic indecomposable modules of a given rank designated the same way; for example, $I_{(10)}$ may stand in for two or more non-isomorphic indecomposable $R_{\mathcal{M}}$-modules of rank $(1,0)$.

With the assumptions above, let M be a finitely generated R-module. To the $R_{\mathcal{M}}$-module $M_{\mathcal{M}}$, we associate the six-tuple $\vec{w} = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{bmatrix} \in \mathcal{N}^6$ to indicate that $M_{\mathcal{M}}$ is a direct sum of generic indecomposable $R_{\mathcal{M}}$-modules of form:

$$M_{\mathcal{M}} \text{ “ = ” } I_{(10)}^{w_1} \oplus I_{(01)}^{w_2} \oplus I_{(11)}^{w_3} \oplus I_{(12)}^{w_4} \oplus I_{(21)}^{w_5} \oplus I_{(22)}^{w_6}. \tag{3.2a}$$

Again, the interpretation of this representation $\vec{w}$ for $M_{\mathcal{M}}$ is that $M_{\mathcal{M}}$ is a direct sum of indecomposable $R_{\mathcal{M}}$-modules, with w_1 indecomposables of rank $(1,0)$, w_2 indecomposables of rank $(0,1)$, etc. As in the one minimal prime case, it does not matter that indecomposables having the same subscript may not be isomorphic, nor that the vector $\vec{w}$ may not be unique.

Now suppose that $M_{\mathcal{M}}$, represented as in the expression above, has $\operatorname{rank}(M_{\mathcal{M}}) = (s,t)$, for $s, t \in \mathcal{N}$, i.e. $\dim_{R_{P_1}}(M_{P_1}) = s$, $\dim_{R_{P_2}}(M_{P_2}) = t$, and $\vec{w} := \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{bmatrix}$ for some $w_1, w_2, w_3, w_4, w_5, w_6 \in \mathcal{N}$. Then we have the following relations, where A is the matrix shown below, the vectors $\vec{a}_i$ are the

columns of A, and $\vec{w}^T$ denotes the transpose of $\vec{w}$:

$$\begin{bmatrix} s \\ t \end{bmatrix} = (\mathrm{rank}(M_{\mathcal{M}})\,)^{\,T} = A\vec{w}^T,$$

$$A := \begin{bmatrix} 1 & 0 & 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 & \vec{a}_4 & \vec{a}_5 & \vec{a}_6 \end{bmatrix},$$

$$\begin{bmatrix} s \\ t \end{bmatrix} = w_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + w_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + w_4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + w_5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + w_6 \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \tag{3.2b}$$

$$\begin{aligned} s &= w_1 + w_3 + w_4 + 2w_5 + 2w_6, \\ t &= w_2 + w_3 + 2w_4 + w_5 + 2w_6. \end{aligned} \tag{3.2c}$$

Notation 3.3. Matrix representation: the three minimal prime case: Suppose $\mathcal{M}$ is a maximal ideal of R containing exactly three minimal primes P_1, P_2 and P_3. Suppose also that every indecomposable $R_{\mathcal{M}}$-module has rank-tuple one of the list in (2.7.3).

In analogy to the one and two minimal prime cases, we represent generic indecomposable $R_{\mathcal{M}}$-modules by the letter I with subscripts that display the rank-tuples, namely,

$$I_{(100)},\ I_{(010)},\ I_{(001)},\ I_{(110)},\ I_{(101)},\ I_{(011)},\ I_{(111)},\ I_{(112)}.$$

Each expression, e.g. $I_{(112)}$, is used for every indecomposable module of that subscript rank.

As in 3.2, we associate the eight-tuple

$$\vec{w} = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 & w_7 & w_8 \end{bmatrix} \in \mathcal{N}^8$$

to $M_{\mathcal{M}}$, if $M_{\mathcal{M}}$ is a direct sum of generic indecomposable $R_{\mathcal{M}}$-modules of form:

$$M_{\mathcal{M}} \text{ “ = ” } I^{w_1}_{(100)} \oplus I^{w_2}_{(010)} \oplus I^{w_3}_{(001)} \oplus I^{w_4}_{(110)} \oplus I^{w_5}_{(101)} \oplus I^{w_6}_{(011)} \oplus I^{w_7}_{(111)} \oplus I^{w_8}_{(112)}. \tag{3.3a}$$

Let $\operatorname{rank}(M_{\mathcal{M}}) = (r, s, t)$, for $r, s, t \in \mathcal{N}$, i.e. $\dim_{R_{P_1}}(M_{P_1}) = r$, $\dim_{R_{P_2}}(M_{P_2}) = s$, $\dim_{R_{P_3}}(M_{P_3}) = t$; we define B to be the matrix shown with columns $\vec{b}_1, \vec{b}_2, \cdots, \vec{b}_8$. Then

$$\begin{bmatrix} r \\ s \\ t \end{bmatrix} = (\operatorname{rank}(M_{\mathcal{M}}))^T = B\vec{w}^T, \quad B := \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} \vec{b}_1 \, \vec{b}_2 \cdots \vec{b}_8 \end{bmatrix},$$

$$\begin{bmatrix} r \\ s \\ t \end{bmatrix} = w_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + w_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + w_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + w_4 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + w_5 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + w_6 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + w_7 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + w_8 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \tag{3.3b}$$

$$\begin{aligned} r &= w_1 + w_4 + w_5 + w_7 + w_8, \\ s &= w_2 + w_4 + w_6 + w_7 + w_8, \\ t &= w_3 + w_5 + w_6 + w_7 + 2w_8 \,. \end{aligned} \tag{3.3c}$$

Next we show that Baeth's Theorem 2.7 for complete local ring-orders holds in the non-complete case.

Theorem 3.4. *Suppose that $(R, \mathcal{M})$ is a local ring-order with finite representation type such that R is equicharacteristic, not of characteristic 2, 3 or 5, with perfect residue field. Let M be an indecomposable torsion-free R-module. Then R has one, two or three minimal prime ideals and the rank of M is given by the following lists:*

(3.4.1) *If R has one minimal prime ideal, then* $\operatorname{rank}(M) \leq 3$.

(3.4.2) *If R has two minimal prime ideals, then* $\operatorname{rank}(M)$ *is one of the following:*

$$(1,0),(0,1),(1,1),(1,2),(2,1),(2,2).$$

(3.4.3) *If R has three minimal prime ideals then* $\operatorname{rank}(M)$ *is one of these:*

$$(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1),(1,1,1),(1,1,2).$$

Proof. We use the notation for the completions $\widehat{R}$ and $\widehat{M}$ from 2.1. The property described in the first sentence of the statement is also true for $\widehat{R}$. Since R has bounded representation type, so does $\widehat{R}$ (by [30, Theorem 1.9 and page 5] and [33, Appendix]). By Dade's Theorem, each of R and $\widehat{R}$ has at most three minimal prime ideals [14, page 410]. We consider four cases of Theorem 3.4 separately in the four propositions below; these cases arise from the possible relationships between minimal prime ideals of R and those of $\widehat{R}$, that is, the number of prime ideals of $\widehat{R}$ that lie over each minimal prime ideal of R. For example, if R has a unique minimal prime P, then there could be one, two or three minimal prime ideals $\widehat{Q}$ of $\widehat{R}$ lying over P, that is, such that $\widehat{Q} \cap R = P$. If R has three minimal prime ideals, then, since $\widehat{R}$ has at most three minimal prime ideals, there can be just one minimal prime ideal of $\widehat{R}$ lying over each of the minimal primes of R etc.

We show in Propositions 3.5 - 3.8 that, in each of the cases above, if M is a non-zero R-module such that the rank of M is not on the appropriate list from Theorem 3.4, then the completion $\widehat{M} = M \otimes_R \widehat{R}$ has a non-trivial $\widehat{R}$-submodule $\widehat{X}$ that is a direct summand of $\widehat{M}$. Moreover, for every minimal prime P of R, the entries of $\operatorname{rank}(\widehat{X})$ corresponding to primes of $\widehat{R}$ lying over P agree. (In case there is no splitting from R to $\widehat{R}$, as in 3.5, the entries of $\operatorname{rank}(\widehat{X})$ need not repeat at all.) Then $\widehat{M} = \widehat{X} \oplus \widehat{Y}$, where $\widehat{Y}$ is another $\widehat{R}$-module such that its rank-entries agree in those coordinates. By Proposition 2.10, $\widehat{X}$ and $\widehat{Y}$ are extended from R-modules X and Y such that $\operatorname{rank}(X)$ and $\operatorname{rank}(Y)$ have the same entries as the ranks of $\widehat{X}$ and $\widehat{Y}$ (although they may have fewer repetitions in their rank-entries). Thus we have $X \neq M$ or (0) and $(X \oplus Y) \otimes \widehat{R} = \widehat{M}$, and so $X \oplus Y = M$. The four propositions complete the proof of Theorem 3.4. □

Proposition 3.5. *Suppose that R is a local ring-order with bounded representation type such that R is equicharacteristic, not of characteristic 2, 3 or 5, with perfect*