HIGHER-ORDER EXPLICIT AND IMPLICIT DYNAMIC TIME INTEGRATION METHODS

By

Colin W. Keierleber

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Higher-Order Explicit and Implicit Dynamic Time

Integration Methods

BY

Colin Walker Keierleber

SUPERVISORY COMMITTEE:

Approved	Date
Signature	April 18, 2003
Dr. Barry Rosson Typed Name	
Signature	4/24/03
Dr. Mehrdad Negahban Typed Name	
Signature	April 18, 2003
Dr. Dean Sicking Typed Name	
Signature	April 18, 2003
Dr. Christopher Tuan Typed Name	
Signature	
Typed Name	
Signature	
Typed Name	

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HIGHER-ORDER EXPLICIT AND IMPLICIT DYNAMIC
TIME INTEGRATION METHODS

Colin Walker Keierleber, Ph.D.

University of Nebraska, 2003

Advisor: Barry T. Rosson

General procedures are presented for the use of higher-order approximations of a system's dynamic response using new time integration schemes. Traditional methods assume a constant or linearly varying acceleration in between discrete time points. Both explicit and implicit methods are developed and presented that make use of higher-order polynomial acceleration variations for single-degree-of-freedom and multi-degree-of-freedom dynamic structural analyses. An implicit method that makes use of a variable θ to increase the stability and accuracy of the method is also presented. Both direct time integration and modal analysis algorithms are developed and implemented in computer programs to investigate their accuracy and usefulness.

The relative merits and stability aspects of the methods are described. The higher-order methods are compared to traditional explicit methods, such as the Central Difference and Runge-Kutta Methods, and traditional implicit methods, such as the Linear Acceleration, Wilson-θ, Newmark, and Houbolt Methods. Amplitude decay and period elongation comparisons are used to measure the accuracy of the methods when applied to a single-degree-of-freedom free vibration problem. The equivalent peak displacement percent difference and percent time difference to peak displacement are used to measure the accuracy of the methods when applied to multi-degree-of-freedom problems.

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1 INTRODUCTION

In many engineering problems, the evaluation of a structure using a static analysis is not sufficient to obtain the true response of the system. In these cases a dynamic analysis may be necessary. The term dynamic means that the loads on the structure and the response (i.e., displacements, velocities, internal stresses, etc.) of the structure vary with time. Due to the variance of the response with time, the inertia and damping forces must also be included in the equation(s), resulting in the equation(s) of motion for the system.

In truth, no loads that are applied to a structure are ever truly static. The most important parameter determining the extent of the dynamic effect each load has on a structure is the natural period of vibration of the structure, T. The natural period of vibration of a structure is the time required for it to complete one full cycle of free vibration. If the duration of the load is large compared to the natural period, then the dynamic effect will be insignificant, and a static analysis will be sufficient. If the duration of the load is of the same magnitude or shorter than the natural period, it will induce a significant dynamic response, and a dynamic analysis will be necessary.

Examples of dynamic problems are numerous. They include the loading of structures due to a collision or explosive blast, the response of a bridge to moving vehicles, or the landing impact upon aircraft. Other dynamic loads are the result of weather or other natural causes, such as wind loads on high-rise structures or wave loadings on off-shore oil-drilling platforms. Two other common dynamic problems are the response of a structure to an earthquake, where the loading takes the form of an acceleration history, and structures subjected to alternating forces caused by oscillating

machinery. In general, all of these examples and the majority of all other dynamic loads are due to one of the following sources: (1) environmental, (2) machine induced, (3) vehicular induced, and (4) blast induced (15).

All dynamic problems can be classified as one of two types, wave propagation problems or inertial problems. Wave propagation problems are those in which the behavior at the wave front is of importance. In this type of problem, the response is dominated by the intermediate and high-frequency modes. Examples of this category include the shock response of a structure from explosive weapons and travel of a pressure wave through a material due to an impact. All other dynamic problems can be considered inertial. The response of these problems is dominated by a small number of low frequency modes. These problems are also often called structural dynamics problems. Examples include the response of a structure to an earthquake or a ramp loading.

In special cases the governing equations of a dynamic problem can be solved exactly, but in most cases a numerical procedure must be used. To solve the problem numerically, the governing differential equations are first discretized in space. This procedure is called semi-discretization. The semi-discretization, done using either finite element or finite difference methods, reduces the problem to a system of ordinary differential equations in time. For structural dynamics problems, this leads to the commonly utilized system of equations

$$[m]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = \{f\} \tag{1.1}$$

where $[m]$, $[c]$, and $[k]$ are the structural mass, damping, and stiffness matrices and $\{\ddot{x}\}$, $\{\dot{x}\}$, and $\{x\}$ are the nodal acceleration, velocity, and displacement vectors. This system of equations is then numerically integrated in order to solve the dynamic problem.

1.1 Direct Time Integration

The process of solving the governing system of second-order differential equations, (1.1), is called direct time integration. The equations are integrated using a step-by-step procedure. The term 'direct' simply means that the integration is carried out without any prior transformation to the equations.

Direct integration is based on two ideas. The first is to impose the equations of motion (balance) at each discrete time point within the interval of solution. After each time step, Δt, the equation of motion, which includes inertial, damping, internal, and applied forces, must be satisfied. Second, a form for the variation of acceleration, velocity, and displacement is assumed within each time interval. These two concepts are used to develop a time stepping process to obtain the dynamic response of the structure. Each different set of assumptions for these variations results in a different direct integration method. These assumptions largely affect the stability and accuracy of the method.

All of the direct integration methods can be categorized as either explicit or implicit methods. These two groups approach the solution of the dynamic problem in a slightly different manner. Explicit methods begin with the equation(s) of motion at time t, while implicit methods begin with the equation(s) of motion at time $t+\Delta t$. These

different approaches lead to advantages and disadvantages that are common to all or most of the methods that fall within each group.

1.2 Explicit Methods

Explicit methods consider the dynamic balance equations at time t to evaluate the solution at time $t+\Delta t$. In other words, the displacement, velocity, and acceleration at time $t+\Delta t$ are all obtained using only the known values at time t and prior discrete time points. These methods are particularly well suited for short duration dynamics problems or wave propagation problems in which the intermediate to high frequency modes of vibration are important to the response.

Explicit methods have many advantages in common. In an explicit scheme, the solution only requires the inversion of the mass matrix or an effective mass matrix, which is often diagonal. Hence, the computational cost per step of an explicit method is often quite low and little storage is required.

Explicit methods also share a common disadvantage. These methods are only conditionally stable. This means that the methods only remain numerically stable for a limited range of time step values. In fact, the step size restriction is often more restrictive than the desired accuracy would otherwise require. So, while the computational cost per step is low, the total computational cost is still high due to the required number of time steps. This is the reason that explicit methods are better suited for problems of short duration. They are not often used to analyze problems of moderate duration such as an earthquake response problem (7).

Possibly the two most common explicit methods are the Central Difference Method and the Fourth-Order Runge-Kutta Method.

1.2.1 Central Difference Method

The second-order Central Difference Method is one of the most commonly used explicit direct integration methods. It is said to have the highest accuracy and the maximum stability limit for any explicit method of order two (7). (Here the order two refers to a second-order displacement. In following chapters, the polynomial order of the assumed acceleration variation will be referred to as the order of the method.) However, like all explicit methods, the Central Difference Method has the disadvantage of requiring very small time steps.

The method is based on the following central difference formulas (7):

$$\{x\}_{i+1/2} = \frac{1}{2}\left(\{x\}_{i+1} + \{x\}_i\right) \tag{1.2.1.1}$$

$$\{x\}_{i-1/2} = \frac{1}{2}\left(\{x\}_i + \{x\}_{i-1}\right) \tag{1.2.1.2}$$

$$\{\dot{x}\}_i = \frac{1}{\Delta t}\left(\{x\}_{i+1/2} - \{x\}_{i-1/2}\right) \tag{1.2.1.3}$$

$$\{\dot{x}\}_{i+1/2} = \frac{1}{\Delta t}\left(\{x\}_{i+1} - \{x\}_i\right) \tag{1.2.1.4}$$

$$\{\dot{x}\}_{i-1/2} = \frac{1}{\Delta t}\left(\{x\}_i - \{x\}_{i-1}\right) \tag{1.2.1.5}$$

and

$$\{\ddot{x}\}_i = \frac{1}{\Delta t}\left(\{\dot{x}\}_{i+1/2} - \{\dot{x}\}_{i-1/2}\right) \tag{1.2.1.6}$$

Using equations (1.2.1.1)-(1.2.1.3) and (1.2.1.4)-(1.2.1.6), the velocities and accelerations are found to be a function of the displacements as

$$\{\dot{x}\}_i = \frac{1}{2\Delta t}\left(\{x\}_{i+1} - \{x\}_{i-1}\right) \tag{1.2.1.7}$$

and

$$\{\ddot{x}\}_i = \frac{1}{\Delta t^2}\left(\{x\}_{i+1} - 2\{x\}_i + \{x\}_{i-1}\right) \tag{1.2.1.8}$$

The error in Eqs. (1.2.1.7) and (1.2.1.8) is of order Δt^2. So, if the time step is halved, the error in the displacement is quartered.

The variations of the velocities and accelerations above are used to develop the Central Difference Method algorithm that is given in Table 1.1. Due to the appearance of the subscript (i-1), a special starter routine must be added to evaluate the first time step. If the mass and damping matrices are diagonal, no matrix inversions are necessary. However, if the damping matrix is not diagonal, the effective mass matrix is not diagonal and a matrix inversion is necessary. This is one of the weak points of the Central Difference Method.

Other versions of the Central Difference Method have been suggested that require an inversion of only the mass matrix. One such method is given in Table 1.2. In this method, the backward difference approximation is used for calculating the displacement at time t. Eq. (1.2.1.7) is then replaced by

$$\{\dot{x}\}_i = \frac{1}{\Delta t}\left(\{x\}_i - \{x\}_{i-1}\right) \tag{1.2.1.9}$$

If the mass matrix is diagonal, no matrix inversion is necessary.

Noor and Lambiotte (11) and Belytschko (3) have presented an alternative algorithm that uses a different set of central difference formulas. This algorithm is given

TABLE 1.1. Central Difference Method Algorithm.

A. Initial inputs :

1. Input the stiffness matrix $[k]$, mass matrix $[m]$, and damping matrix $[c]$.
2. Select a time step Δt.
3. Input the inital displacement and velocity vectors, $\{x\}_0$ and $\{\dot{x}\}_0$.

B. Initial calculations :

1. Calculate the initial acceleration.

$$\{\ddot{x}\}_0 = [m]^{-1}\{\{F(0)\} - [c]\{\dot{x}\}_0 - [k]\{x\}_0\}$$

2. Calculate the integration constants.

$$a_0 = \frac{1}{\Delta t^2} \qquad a_1 = \frac{1}{2\Delta t}$$

$$a_2 = 2a_0 \qquad a_3 = \frac{1}{a_2}$$

3. Calculate the displacement vector at time -Δt.

$$\{x\}_{-\Delta t} = \{x\}_0 - \Delta t\{\dot{x}\}_0 + a_3\{\ddot{x}\}_0$$

4. Form the effective mass matrix.

$$[\hat{m}] = a_0[m] + a_1[c]$$

C. For each time step :

1. Calculate the effective load vector at time t.

$$\{\hat{F}\}_t = \{F\}_t - ([k] - a_2[m])\{x\}_t - (a_0[m] - a_1[c])\{x\}_{t-\Delta t}$$

2. Solve for the displacement at time $t + \Delta t$.

$$\{x\}_{t+\Delta t} = [\hat{m}]^{-1}\{\hat{F}\}_t$$

3. Calculate the acceleration and velocity vectors at time t.

$$\{\ddot{x}\}_t = a_0(\{x\}_{t-\Delta t} - 2\{x\}_t + \{x\}_{t+\Delta t})$$

$$\{\dot{x}\}_t = a_1(-\{x\}_{t-\Delta t} + \{x\}_{t+\Delta t})$$

TABLE 1.2. Alternative Central Difference Method Algorithm.

A. Initial inputs :

1. Input the stiffness matrix $[k]$, mass matrix $[m]$, and damping matrix $[c]$.
2. Select a time step Δt.
3. Input the inital displacement and velocity vectors, $\{x\}_0$ and $\{\dot{x}\}_0$.

B. Initial calculations :

1. Calculate the initial acceleration.

$$\{\ddot{x}\}_0 = [m]^{-1}\{\{F(0)\} - [c]\{\dot{x}\}_0 - [k]\{x\}_0\}$$

2. Calculate the displacement vector at time -Δt.

$$\{x\}_{-\Delta t} = \{x\}_0 - \Delta t\{\dot{x}\}_0 + \frac{\Delta t^2}{2}\{\ddot{x}\}_0$$

C. For each time step :

1. Calculate the effective load vector at time t.

$$\{\hat{F}\}_t = \{F\}_t - [c]\{\dot{x}\}_t - [k]\{x\}_t$$

2. Solve for the accleration at time t.

$$\{\ddot{x}\}_t = [m]^{-1}\{\hat{F}\}_t$$

3. Calculate the velocity and displacement vectors at time $t + \Delta t$.

$$\{x\}_{t+\Delta t} = -\{x\}_{t-\Delta t} + 2\{x\}_t + \Delta t^2\{\ddot{x}\}_t$$

$$\{\dot{x}\}_{t+\Delta t} = \frac{1}{\Delta t}(\{x\}_{t+\Delta t} - \{x\}_t)$$

in Table 1.3. In this method, instead of using Eqs. (1.2.1.7) and (1.2.1.8), the following rearranged forms of Eqs. (1.2.1.6) and (1.2.1.4) are used.

$$\{\dot{x}\}_{i+1/2} = \{\dot{x}\}_{i-1/2} + \Delta t\{\ddot{x}\}_i \tag{1.2.1.10}$$

and

$$\{x\}_{i+1} = \{x\}_i + \Delta t\{\dot{x}\}_{i+1/2} \tag{1.2.1.11}$$

Unfortunately, this method still requires the inversion of an effective mass matrix that contains the damping matrix.

The largest disadvantage to the Central Difference Method is its conditional stability. The stability limit has been shown to be

$$\Delta t \le \Delta t_{cr};\ \Delta t_{cr} = \frac{2}{\omega_{max}} \tag{1.2.1.12}$$

for an undamped structure, where ω_{max} is the maximum natural frequency of the structure (7). In practical problems, Δt must be roughly 25% less than Δt_{cr} because of round-off errors. Then, if the damping is nondiagonal, the stability limit is reduced even further. Belytschko (3) suggests a value in the range of 5-20% of Δt_{cr} be used.

1.2.2 Fourth-Order Runge-Kutta Method

The fourth-order Runge-Kutta Method is another commonly used explicit method (7). The algorithm for this method is shown in Table 1.4. The variation equations are as follows:

$$\{x\}_{i+1} = \{x\}_i + \frac{\Delta t}{6}\left(\{\dot{x}\}_1 + \{\dot{x}\}_2 + \{\dot{x}\}_3 + \{\dot{x}\}_4\right) \tag{1.2.2.1}$$

and