HILBERT-SAMUEL POLYNOMIALS AND BUILDING INDECOMPOSABLE MODULES

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Let $(R, \mathfrak{m}, \mathsf{k})$ be a Noetherian local ring and M and N be finitely generated R-modules. In this thesis, we give precise formulas for the generalized Hilbert-Samuel polynomials associated to the torsion and contravariant extension functors, that is, polynomials giving the lengths of the modules $\mathrm{Tor}_i^R(M, N/\mathfrak{m}^n N)$ and $\mathrm{Ext}_R^i(M, N/\mathfrak{m}^n N)$, respectively. One application of these results is that they can be used to give information about the dimensions of syzygies of finite length modules.

We also show that if R is complete and has depth at least 2, then one can build indecomposable modules of arbitrarily prescribed constant rank. Moreover, if R is assumed to be Cohen-Macaulay, then these modules can be chosen to be maximal Cohen-Macaulay when localized on the punctured spectrum.

Contents

Chapter 1

Generalized Hilbert-Samuel polynomials

Let $(R, \mathfrak{m}, \mathsf{k})$ be a local Noetherian ring with Krull dimension d, maximal ideal $\mathfrak{m}$ and residue field k. Throughout the thesis, all modules will be assumed to be finitely generated. Let $I \subseteq R$ be an ideal and M and N be R-modules. We begin by considering the following function associated to the R-module N:

$$n \mapsto \lambda(N/\mathfrak{m}^n N),$$

where $\lambda(-)$ denotes the length of a composition series for an R-module. In a 1947 paper [20], Pierre Samuel showed that this function is given by a polynomial $H_N(n)$ for n large. This polynomial is commonly referred to as the Hilbert-Samuel polynomial for N and has been an invaluable tool in commutative algebra for more than a half-century. For instance, the degree of the polynomial $H_N(n)$ is precisely the Krull dimension of N, a result which is a cornerstone of dimension theory.

In the 1990s, Vijay Kodiyalam was interested in how other well-known module invariants of $N/\mathfrak{m}^n N$ grow as one varies n [15]. We mention four such module invariants here: (1) projective dimension $\mathrm{pd}_R(-)$, (2) injective dimension $\mathrm{id}_R(-)$, (3) the i^{th} Betti number $\beta_i^R(-)$, and (4) the i^{th} Bass number $\mu_R^i(-)$. For an R-module L, $\mathrm{pd}_R(L)$ is equal to the length of a minimal free resolution of L, whereas $\beta_i^R(L)$ is equal to the rank of the i^{th} free module in such a free resolution of L. Analogously, $\mathrm{id}_R(L)$ is equal to the length of an injective resolution of L, and $\mu_R^i(L)$ is equal to the number of copies of the injective hull of k appearing as direct summands in the i^{th} module of such an injective resolution.

The following theorem of Kodiyalam is quite general and lends insight into the growth of these four module invariants. We will write $\mathcal{R} = \bigoplus_{n \geq 0} I^n$ to denote the Rees ring of I and $\mathcal{N} = \bigoplus_{n \geq 0} I^n N$ to denote the graded $\mathcal{R}$-module associated with the I-adic filtration on N.

Theorem 1.0.1 (Kodiyalam, [15])**.** *If $\lambda(N \otimes_R M) < \infty$, then, for all large n, each*

of the functions:

$$n \mapsto \lambda(\mathrm{Tor}_i^R(M, N/I^nN)),$$
$$n \mapsto \lambda(\mathrm{Ext}_R^i(M, N/I^nN))$$

is a polynomial in n with degree at most $\max\{0, \dim_{\mathcal{R}}(\mathcal{N} \otimes_R M) - 1\}$.

This theorem is proven using the Hilbert-Serre Theorem (see [18], for instance) and a technical proposition concerning long exact sequences of graded $\mathcal{R}$-modules. As remarked above, Theorem 1.0.1 yields information about the growth of Betti numbers, Bass numbers, etc.. In the following corollary, $\ell_N(I) := \dim_{\mathcal{R}}(\mathcal{N}/\mathfrak{m}\mathcal{N})$ will denote the analytic spread of I on N.

Corollary 1.0.2 (Kodiyalam, [15])**.** *For any fixed non-negative integer i and for all n sufficiently large, $\beta_i^R(N/I^nN)$ and $\mu_R^i(N/I^nN)$ are polynomials in n of degree at most* $\max\{0, \ell_N(I) - 1\}$.

Proof. Since $\beta_i^R(N/I^nN)$ is equal to $\lambda(\mathrm{Tor}_i^R(\mathsf{k}, N/I^nN))$ and $\mu_R^i(N/I^nN)$ is equal to $\lambda(\mathrm{Ext}_R^i(\mathsf{k}, N/I^nN))$, the result follows immediately from Theorem 1.0.1 with $M = \mathsf{k}$. □

Corollary 1.0.3 (Kodiyalam, [15])**.** *For large n, both* $\mathrm{pd}_R(N/I^nN)$ *and* $\mathrm{id}_R(N/I^nN)$ *attain stable constant values.*

Additionally, Kodiyalam used Theorem 1.0.1 to give a new proof of the following result of Markus Brodmann: if J is a proper ideal of R, then, for all large n, $\mathrm{depth}_J(N/I^nN)$ attains a stable constant value [2].

In 2002, Emanoil Theodorescu improved Theorem 1.0.1 by relaxing the conditions on the modules M and N. That is, rather than assuming that $N \otimes_R M$ has finite length, we need only assume that the modules $\mathrm{Tor}_i^R(M, N/I^nN)$, respectively $\mathrm{Ext}_R^i(M, N/I^nN)$, have finite length for large n.

Theorem 1.0.4 (Theodorescu, [23])**.** *Suppose that* $\mathrm{Tor}_i^R(M, N/I^nN)$ *has finite length for $n \gg 0$. Then for large n the function:*

$$n \mapsto \lambda(\mathrm{Tor}_i^R(M, N/I^nN))$$

is given by a polynomial, denoted $\tau_i^{M,N,I}(n)$. Moreover

$$\deg(\tau_i^{M,N,I}(n)) \leq \max\{\dim(\mathrm{Tor}_i^R(M, N/I^nN)), \ell_N(I) - 1\},$$

and equality holds, if $\dim(\mathrm{Tor}_i^R(M, N/I^nN)) \geq \ell_N(I)$.

Similarly, suppose that $\mathrm{Ext}_R^i(M, N/I^nN)$ *has finite length for $n \gg 0$. Then for large n the function:*

$$n \mapsto \lambda(\mathrm{Ext}_R^i(M, N/I^nN))$$

is given by a polynomial, denoted $\epsilon_{M,N,I}^i(n)$. Further,

$$\deg(\epsilon_{M,N,I}^i(n)) \leq \max\{\dim(\mathrm{Ext}_R^i(M, N/I^nN)), \ell_N(I) - 1\},$$

and again, if $\dim(\operatorname{Ext}^i_R(M, N/I^nN)) \geq \ell_N(I)$*, then equality holds.*

The precise degrees of these Hilbert-Samuel polynomials, $\tau_i^{M,N,I}(n)$ and $\epsilon^i_{M,N,I}(n)$, are not easy to determine. Determining these degrees has been the principal interest of recent research on these polynomials, and by making various assumptions on M, N, I and/or R, some progress has been made in this regard (see [23], [12], and [13]). One of the purposes of this chapter is to improve the known estimates for the degrees of the polynomials $\tau_i^{M,N,I}(n)$ and $\epsilon^i_{M,N,I}(n)$ in the case $I = \mathfrak{m}$ by giving a precise formula for these degrees. Previous results in this case for the torsion functor were given in [12] and [13], where various assumptions were made in order to control this degree. We do not need to make any assumptions on M, N or R to obtain these formulas, and we need only make modest assumptions on them to obtain a formula that makes direct reference only to M and N.

In Section 1.1, we begin by giving a general formula (see Proposition 1.1.2) for the degree of the Hilbert-Samuel polynomial associated to general cohomology or homology modules, which specializes to our main results in Section 1.2, when the ideal in question is $\mathfrak{m}$ and the cohomology is determined by either the contravariant extension functor (Theorem 1.2.3) or the torsion functor (Theorem 1.2.6).

As we will see below, the degree of $\epsilon^i_{M,N,\mathfrak{m}}(n)$ is partially controlled by the dimension of the i^{th} syzygy of M. Consequently, as an application of the degree formula, we prove the following proposition in Section 1.4, which yields some information about the dimensions of the syzygies of finite length modules. We denote the i^{th} syzygy of M by $\Omega^i_R(M)$.

Proposition 1.0.5. *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a local ring and* M *a finitely generated* R*-module, free of constant rank on the punctured spectrum of* R*. Assume* $\dim(R) \geq 2$*, that the Betti numbers of* M *are non-decreasing and that* $i < \operatorname{pd}_R(M)$*. Then* $\dim_R(\Omega^{i+1}_R(M)) = d$*.*

Another purpose of this chapter, especially in regards to Sections 1.2 and 1.3, is to lay the groundwork for results concerning indecomposable modules in Chapter 2, where knowledge of the relative growth of the Hilbert-Samuel polynomials involving large syzygies of the residue field k is required.

Finally, in Section 1.5, we show that the results of Sections 1.2 and 1.3 can, in many cases, be extended to more general filtrations to give results for the extension functor parallel to those given in [13] for the torsion functor.

The work in this chapter is joint with Daniel Katz, Janet Striuli and Emanoil Theodorescu.

1.1 General homology

Throughout, $(R, \mathfrak{m}, \mathsf{k})$ will denote a Noetherian local ring of Krull dimension d and all modules will be finitely generated R-modules. Again, the principal goal of this

chapter is to provide a formula for the degrees of the Hilbert-Samuel polynomials for the torsion and contravariant extension functors. In this section, we prove a general result about the Hilbert-Samuel polynomial associated to an ideal and the homology of a complex. We start by letting $I \subseteq R$ and

$$\mathcal{C}: \quad X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$$

be a complex of finitely generated R-modules with $Y \neq 0$. We will be interested in the complexes resulting from applying the functor $- \otimes R/I^n$ to $\mathcal{C}$:

$$\mathcal{C} \otimes R/I^n: \quad X \otimes R/I^n \xrightarrow{\alpha \otimes R/I^n} Y \otimes R/I^n \xrightarrow{\beta \otimes R/I^n} Z \otimes R/I^n,$$

and we will assume throughout that I is such that the homology modules $\mathrm{H}(\mathcal{C} \otimes R/I^n) := \ker(\beta \otimes R/I^n)/\operatorname{im}(\alpha \otimes R/I^n)$ associated to $\mathcal{C} \otimes R/I^n$ have finite length for n large. In a 2002 paper, Emanoil Theodorescu proved that the growth of the lengths of these homology modules can be given by a polynomial. (The result comes as a consequence of his Proposition 3 in [23].)

Proposition 1.1.1 (Theodorescu, [23])**.** *The lengths of the modules* $\mathrm{H}(\mathcal{C} \otimes R/I^n)$ *are given by a polynomial* $P_I^{\mathcal{C}}(n)$ *for* n *large. Moreover,*

$$\deg(P_I^{\mathcal{C}}(n)) \leq \max\{\dim(\mathrm{H}(\mathcal{C})),\ \ell_R(I) - 1\}.$$

If $\dim(\mathrm{H}(\mathcal{C})) \geq \ell_R(I)$*, then equality holds.*

The following proposition strengthens Proposition 1.1.1, in that the degree of the polynomial $P_I^{\mathcal{C}}(n)$ is given precisely. We set

$$\mathcal{M} := \bigoplus_{n \geq 0} (I^n Z \cap \operatorname{im}(\beta))/I^n \operatorname{im}(\beta).$$

Note that $\mathcal{M}$ is a finitely generated graded module over the Rees ring of R with respect to I, so that if its graded components have finite length as R-modules, then these lengths are ultimately given by a rational polynomial of degree $\dim_{\mathcal{R}}(\mathcal{M}) - 1$.

Proposition 1.1.2. *Let* $(R, \mathfrak{m}, \mathsf{k})$ *be a local ring and* $\mathcal{C}$ *be a complex of finitely generated* R*-modules,*

$$\mathcal{C}: \quad X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z,$$

with $Y \neq 0$*. Let* $I \subseteq R$ *be an ideal such that the lengths of the homology modules* $\mathrm{H}(\mathcal{C} \otimes R/I^n)$ *are finite for* n *large and let* $P_I^{\mathcal{C}}(n)$ *denote the corresponding Hilbert-Samuel polynomial. Then*

$$\deg(P_I^{\mathcal{C}}(n)) = \max\{\dim(\mathrm{H}(\mathcal{C})),\ \dim_{\mathcal{R}}(\mathcal{M}) - 1\}.$$

Proof. Consider the complex $\mathcal{C} \otimes R/I^n$:

$$\mathcal{C} \otimes R/I^n : \quad X \otimes R/I^n \overset{\alpha\otimes R/I^n}{\longrightarrow} Y \otimes R/I^n \overset{\beta\otimes R/I^n}{\longrightarrow} Z \otimes R/I^n.$$

By the Artin-Rees Lemma, there exists $h > 0$ so that for $n \geq h$,

$$I^n Z \cap \operatorname{im}(\beta) = I^{n-h}(I^h Z \cap \operatorname{im}(\beta)).$$

Notice that this implies

$$\beta^{-1}(I^n Z \cap \operatorname{im}(\beta)) = \beta^{-1}(I^{n-h}(I^h Z \cap \operatorname{im}(\beta))) = \ker(\beta) + I^{n-h}\beta^{-1}(I^h Z \cap \operatorname{im}(\beta)).$$

Thus it follows that for $n \geq h$,

$$\mathrm{H}(\mathcal{C}\otimes R/I^n) = \frac{\ker(\beta \otimes R/I^n)}{\operatorname{im}(\alpha \otimes R/I^n)} \cong \frac{\beta^{-1}(I^n Z \cap \operatorname{im}(\beta))}{\operatorname{im}(\alpha) + I^n Y} = \frac{\ker(\beta) + I^{n-h}\beta^{-1}(I^h Z \cap \operatorname{im}(\beta))}{\operatorname{im}(\alpha) + I^n Y}.$$

For convenience, we set $A := \operatorname{im}(\alpha)$, $B := \ker(\beta)$, $C := \beta^{-1}(I^h Z)$ and $D := I^h Y$, so we may write

$$\mathrm{H}(\mathcal{C} \otimes R/I^n) \cong \frac{B + I^{n-h}C}{A + I^{n-h}D}.$$

Now for n large,

$$P_I^{\mathcal{C}}(n) = \lambda\left(\frac{B + I^{n-h}C}{A + I^{n-h}D}\right) = \lambda\left(\frac{B + I^{n-h}C}{B + I^{n-h}D}\right) + \lambda\left(\frac{(B + I^{n-h}D)}{(A + I^{n-h}D)}\right) \tag{1.1}$$

$$= \lambda\left(\frac{B + I^{n-h}C}{B + I^{n-h}D}\right) + \lambda\left(\frac{U + I^{n-h}W}{I^{n-h}W}\right), \tag{1.2}$$

where $U := B/A = \mathrm{H}(\mathcal{C})$ and $W := (D + A)/A$. We first note that by [23], Lemma 2, both length expressions on the right hand side of the displayed equations (1.1) and (1.2) are given by polynomials. Let $P_1(n)$ denote the polynomial giving the lengths of $(B + I^{n-h}C)/(B + I^{n-h}D)$ and $P_2(n)$ denote the polynomial giving the lengths of $(U + I^{n-h}W)/I^{n-h}W \cong U/(U \cap I^{n-h}W)$.

We first calculate the degree of $P_1(n)$. For this, we note that

$$\frac{B + I^{n-h}C}{B + I^{n-h}D} \cong \frac{I^n Z \cap \operatorname{im}(\beta)}{I^n \operatorname{im}(\beta)}, \tag{1.3}$$

via the map sending a representative $\bar{b}$ to $\overline{\beta(b)}$ (See Lemma 3.0.1 in the Appendix). Thus, by definition of $\mathcal{M}$, $\deg(P_1(n)) = \dim_{\mathcal{R}}(\mathcal{M}) - 1$.

We now show that the degree of $P_2(n)$ equals $\dim(U) = \dim(\mathrm{H}(\mathcal{C}))$. Since $P_I^{\mathcal{C}}(n) = P_1(n) + P_2(n)$, this will complete the proof of the proposition. For this, note that by